Problem 11639
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Evaluate

$$\int_0^{\pi/2} (\log(2\sin x))^2\, dx.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

First we note that

$$\begin{aligned}\int_0^{\pi/2} (\log(2\sin x))^2\, dx &= \int_0^{\pi/2} (\log 2 + \log(\sin x))^2\, dx \\ &= \frac{\pi \log(2)^2}{2} + 2\log 2 \int_0^{\pi/2} \log(\sin x)\, dx + \int_0^{\pi/2} (\log(\sin x))^2\, dx.\end{aligned}$$

Let $t = \sin^2 x$ then

$$\int_0^{\pi/2} \log(\sin x)\, dx = \frac{1}{2}\int_0^1 \frac{\log(\sqrt{t})}{\sqrt{t}\sqrt{1-t}}\, dt = \frac{1}{4}\int_0^1 (\log t) t^{-1/2}(1-t)^{-1/2}\, dt = \frac{1}{4}\,\frac{d}{dx}(B(x+1/2,1/2)\bigg|_{x=0},$$

and

$$\int_0^{\pi/2} (\log(\sin x))^2\, dx = \frac{1}{2}\int_0^1 \frac{(\log(\sqrt{t}))^2}{\sqrt{t}\sqrt{1-t}}\, dt = \frac{1}{8}\int_0^1 (\log t)^2 t^{-1/2}(1-t)^{-1/2}\, dt = \frac{1}{8}\,\frac{d^2}{dx}(B(x+1/2,1/2)\bigg|_{x=0},$$

where $B(x,y)$ is the Beta function

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}, dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

Now, since the expansion of $x\Gamma(x)$ at zero is

$$\begin{aligned}x\Gamma(x) &= e^{-\gamma x}\prod_{n=1}^{\infty}\left(1+\frac{x}{n}\right)^{-1} e^{x/n} = \left(1-\gamma x + \frac{\gamma^2 x^2}{2} + o(x^2)\right)\prod_{n=1}^{\infty}\left(1+\frac{x^2}{2n^2}+o(x^2)\right) \\ &= 1-\gamma x + \left(\frac{\gamma^2}{2} + \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^2}\right)x^2 + o(x^2) = 1-\gamma x + \left(\frac{\gamma^2}{2}+\frac{\pi^2}{12}\right)x^2 + o(x^2),\end{aligned}$$

and $\Gamma(1/2)^2 = \pi$, it follows that

$$\begin{aligned}B(x+1/2,1/2) &= \frac{\Gamma(x+1/2)\Gamma(1/2)}{\Gamma(x+1)} = \frac{(2x)\Gamma(2x)\Gamma(1/2)^2}{4^x(x\Gamma(x))^2} \\ &= \pi - 2\pi\log(2)x + \left(2\pi\log(2)^2 + \frac{\pi^3}{6}\right)x^2 + o(x^2).\end{aligned}$$

Therefore

$$\frac{d}{dx}(B(x+1/2,1/2)|_{x=0} = -2\pi\log(2), \quad \text{and} \quad \frac{d^2}{dx}(B(x+1/2,1/2)|_{x=0} = 4\pi\log(2)^2 + \frac{\pi^3}{3},$$

which implies that

$$\int_0^{\pi/2} (\log(2\sin x))^2\, dx = \frac{\pi\log(2)^2}{2} + 2\log 2\left(\frac{-\pi\log(2)}{2}\right) + \left(\frac{\pi\log(2)^2}{2} + \frac{\pi^3}{24}\right) = \frac{\pi^3}{24}.$$

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