Problem 11637
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Proposed by Ovidiu Furdui (Romania).

Let $m \geq 1$ be a nonnegative integer. Prove that

$$\int_0^1 \left\{\frac{1}{x}\right\}^m x^m \, dx = 1 - \frac{1}{m+1}\sum_{k=1}^{m} \zeta(k+1).$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

If $n \leq 1/x < n+1$ then

$$\left\{\frac{1}{x}\right\} = \frac{1}{x} - \left\lfloor\frac{1}{x}\right\rfloor = \frac{1}{x} - n.$$

Hence

$$\begin{aligned}
\int_0^1 \left\{\frac{1}{x}\right\}^m x^m \, dx &= \lim_{N\to\infty}\sum_{n=1}^{N}\int_{\frac{1}{n+1}}^{\frac{1}{n}} \left(\frac{1}{x}-n\right)^m x^m dx = \sum_{n=1}^{\infty}\int_{\frac{1}{n+1}}^{\frac{1}{n}} (1-nx)^m dx \\
&= \sum_{n=1}^{\infty}\left[\frac{(1-nx)^{m+1}}{(m+1)n}\right]_{\frac{1}{n}}^{\frac{1}{n+1}} = \frac{1}{m+1}\sum_{n=1}^{\infty}\frac{1}{(n+1)^{m+1}n} \\
&= \frac{1}{m+1}\sum_{n=2}^{\infty}\frac{1}{n^{m+1}(n-1)}.
\end{aligned}$$

On the other hand

$$\begin{aligned}
1 - \frac{1}{m+1}\sum_{k=1}^{m}\zeta(k+1) &= 1 - \frac{1}{m+1}\sum_{k=1}^{m}\sum_{n=1}^{\infty}\frac{1}{n^{k+1}} = 1 - \frac{1}{m+1}\sum_{n=1}^{\infty}\sum_{k=1}^{m}\frac{1}{n^{k+1}} \\
&= \frac{1}{m+1} - \frac{1}{m+1}\sum_{n=2}^{\infty}\sum_{k=1}^{m}\frac{1}{n^{k+1}} \\
&= \frac{1}{m+1} - \frac{1}{m+1}\sum_{n=2}^{\infty}\frac{n^m-1}{n^{m+1}(n-1)} \\
&= \frac{1}{m+1} - \frac{1}{m+1}\sum_{n=2}^{\infty}\frac{1}{n(n-1)} + \frac{1}{m+1}\sum_{n=2}^{\infty}\frac{1}{n^{m+1}(n-1)} \\
&= \frac{1}{m+1}\sum_{n=2}^{\infty}\frac{1}{n^{m+1}(n-1)}.
\end{aligned}$$

□