Problem 11633
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Proposed by Anthony Sofo (Australia).

For real a*, let* $H_n(a) = \sum_{j=1}^{n} 1/j^a$*. Show that for integers* a*,* b*, and* n *with* $a \geq 1$*,* $b \geq 0$*, and* $n \geq 1$*,*

$$\sum_{k=1}^{n} \frac{H_k(1)^2 + H_k(2)}{(k+b)^a} + 2\sum_{k=1}^{n} \frac{H_k(1)H_{k+b-1}(a)}{k} = H_{n+b}(a)(H_n(1)^2 + H_n(2)).$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Let L_n and R_n be the right-hand side and the left-side hand respectively. Then

$$L_1 = \frac{2}{(1+b)^a} + 2H_b(a) = 2H_{1+b}(a) = R_1.$$

Moreover for $n > 1$

$$L_n - L_{n-1} = \frac{H_n(1)^2}{(n+b)^a} + \frac{H_n(2)}{(n+b)^a} + \frac{2H_n(1)H_{n+b-1}(a)}{n}$$

and

$$\begin{aligned} R_n - R_{n-1} &= H_{n+b}(a)(H_n(1)^2 + H_n(2)) - H_{n-1+b}(a)(H_{n-1}(1)^2 + H_{n-1}(2)) \\ &= \left(H_{n+b-1}(a) + \frac{1}{(n+b)^a}\right)(H_n(1)^2 + H_n(2)) \\ &\qquad - H_{n+b-1}(a)\left(\left(H_n(1) - \frac{1}{n}\right)^2 + \left(H_n(2) - \frac{1}{n^2}\right)\right) \\ &= \frac{H_n(1)^2}{(n+b)^a} + \frac{H_n(2)}{(n+b)^a} + \frac{2H_n(1)H_{n+b-1}(a)}{n}. \end{aligned}$$

Since $L_n - L_{n-1} = R_n - R_{n-1}$, the identity follows by induction. □