Problem 11629
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Proposed by Olivier Oloa (France).

Let

$$f(\sigma)=\int_0^1 x^\sigma\left(\frac{1}{\log(x)}+\frac{1}{1-x}\right)^2 dx..$$

(a) Show that $f(0)=\log(2\pi)-3/2$.
(b) Find a closed form expression for $f(\sigma)$ for $\sigma>0$.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

For $n\geq m$ and $\sigma>0$ let

$$F_{n,m}(\sigma)=\int_0^1 \frac{x^\sigma \log(x)^n}{(1-x)^m}\,dx.$$

Then it is known that

$$F_{0,0}(\sigma)=\frac{1}{\sigma+1}\,,\qquad F_{1,1}(\sigma)=\frac{d}{d\sigma}\int_0^1\frac{1-x^\sigma}{1-x}\,dx=(\Psi(\sigma+1)+\gamma)'=-\Psi'(\sigma+1)=-\Psi'(\sigma)+\frac{1}{\sigma^2}$$

where $\Psi(\sigma)=\Gamma'(\sigma)/\Gamma(\sigma)$. For $n>1$, by integration by parts

$$\begin{aligned}F_{n,n}(\sigma)&=\frac{1}{n-1}\left[\frac{x^\sigma\log(x)^n}{(1-x)^{n-1}}\right]_0^1-\frac{1}{n-1}\int_0^1\frac{\sigma x^{\sigma-1}\log(x)^n+nx^{\sigma-1}\log(x)^{n-1}}{(1-x)^{n-1}}\,dx\\&=0-\frac{\sigma F_{n,n-1}(\sigma-1)+nF_{n-1,n-1}(\sigma-1)}{n-1}=-\frac{\sigma F'_{n-1,n-1}(\sigma-1)+nF_{n-1,n-1}(\sigma-1)}{n-1}\end{aligned}$$

This recurrence relation implies that

$$F_{2,2}(\sigma)=\sigma\Psi''(\sigma)+2\Psi'(\sigma)\,,\qquad F_{3,3}(\sigma)=-\binom{\sigma}{2}\Psi'''(\sigma)-3(\sigma-1/2)\Psi''(\sigma)-3\Psi'(\sigma).$$

Hence

$$\begin{aligned}f''(\sigma)&=\int_0^1 x^\sigma\left(\frac{1}{\log(x)}+\frac{1}{1-x}\right)^2\log(x)^2\,dx\\&=F_{0,0}(\sigma)+2F_{1,1}(\sigma)+F_{2,2}(\sigma)\\&=\frac{1}{\sigma+1}-2\Psi'(\sigma)+\frac{2}{\sigma^2}+\sigma\Psi''(\sigma)+2\Psi'(\sigma).\end{aligned}$$

Therefore,

$$f'(\sigma)=\int f''(\sigma)\,d\sigma=\log(\sigma+1)-2\Psi(\sigma)-\frac{2}{\sigma}+\sigma\Psi'(\sigma)+\Psi(\sigma)+a$$

and

$$\begin{aligned}f(\sigma)=\int f'(\sigma)\,d\sigma&=(\sigma+1)\log(\sigma+1)-(\sigma+1)-2\log(\Gamma(\sigma))-2\log(\sigma)+\sigma\Psi(\sigma)+a\sigma+b\\&=(\sigma+1)\log(\sigma+1)-2\log(\Gamma(\sigma+1))+\sigma\Psi(\sigma+1)+(a-1)\sigma+b-2\end{aligned}$$

for some constants a,b.
As $\sigma\to+\infty$ it is easy to verify that $f(\sigma)$ goes to zero. On the other hand

$$f(\sigma)=(1+a)\sigma-1/2-\log(2\pi)+b+O(1/\sigma)$$

which implies that $a = -1$ and $b = \log(2\pi) + 1/2$. Finally

$$f(\sigma) = (\sigma + 1)\log(\sigma + 1) - 2\log(\Gamma(\sigma + 1)) + \sigma\Psi(\sigma + 1) - 2\sigma + \log(2\pi) - 3/2$$

and, since $f(\sigma)$ is continuous, we have that

$$f(0) = \log(2\pi) - 3/2.$$

□

Note that by a similar method, and by using $F_{3,3}(\sigma)$, it is possible to prove that

$$\int_0^1 \left(\frac{1}{\log(x)} + \frac{1}{1-x}\right)^2 \frac{\log(x)}{1-x}\, dx = -5/4 - \gamma/2 + \log(2\pi)/2.$$