Problem 11623
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A fair coin is tossed n times and the results recorded as a bit string. A run is a maximal subsequence of (possibly just one) identical tosses. Let the random variable X_n be the number of runs in the bit string not immediately followed by a longer run. (For instance, with bit string 1001101110, there are six runs, of lengths 1, 2, 2, 1, 3, and 1. Of these, the 2nd, 3rd, 5th, and 6th are not followed by a longer run, so $X_{10} = 4$.) Find $E(X_n)$.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Let

$$a_n = 2^n E(X_n) = \sum_{\mathbf{v}\in\{0,1\}^n} X_n(\mathbf{v}),$$

where $\mathbf{v}$ is a bit string of length n. Then, it easy to verify that the first terms of this sequence are:

$$2, 6, 14, 34, 78, 178, 398, 882, 1934, 4210, \ldots.$$

We denote by 0^a and 1^a respectively a run of 0s or 1s of length $a \geq 1$.
It follows that as $\mathbf{v}$ runs over the 2^n possible choises for $n \geq 2$, we have that

(1) if $\mathbf{v} = 0^n$ then $X_{n+1}(\mathbf{v}\,0) = X_n(\mathbf{v})$ and $X_{n+1}(\mathbf{v}\,1) = X_n(\mathbf{v}) + 1$,

(2) if $\mathbf{v} = \mathbf{w}\,1^j\,0^k$ with $1 \leq j < k$ or $1 \leq k < j$ then

$$X_{n+1}(\mathbf{v}\,0) = X_n(\mathbf{v}) \qquad \text{and} \qquad X_{n+1}(\mathbf{v}\,1) = X_n(\mathbf{v}) + 1,$$

(3) if $\mathbf{v} = \mathbf{w}\,1^k\,0^k$ with $k \geq 1$ then $X_{n+1}(\mathbf{v}\,0) = X_n(\mathbf{v}) - 1$ and $X_{n+1}(\mathbf{v}\,1) = X_n(\mathbf{v}) + 1$,

(4) if $\mathbf{v} = \mathbf{w}\,0^k\,1^k$ with $k \geq 1$ then $X_{n+1}(\mathbf{v}\,0) = X_n(\mathbf{v}) + 1$ and $X_{n+1}(\mathbf{v}\,1) = X_n(\mathbf{v}) - 1$,

(5) if $\mathbf{v} = \mathbf{w}\,0^j\,1^k$ with $1 \leq j < k$ or $1 \leq k < j$ then

$$X_{n+1}(\mathbf{v}\,0) = X_n(\mathbf{v}) + 1 \qquad \text{and} \qquad X_{n+1}(\mathbf{v}\,1) = X_n(\mathbf{v}),$$

(6) if $\mathbf{v} = 1^n$ then $X_{n+1}(\mathbf{v}\,0) = X_n(\mathbf{v}) + 1$ and $X_{n+1}(\mathbf{v}\,1) = X_n(\mathbf{v})$.

Therefore, for $n \geq 2$,

$$\begin{aligned} a_{n+1} &= \sum_{\mathbf{v}\in\{0,1\}^n} X_{n+1}(\mathbf{v}\,0) + \sum_{\mathbf{v}\in\{0,1\}^n} X_{n+1}(\mathbf{v}\,1) \\ &= 2a_n + 2^n - \sum_{k=1}^{\lfloor n/2\rfloor - 1} 2^{n-2k} - 2 = 2a_n + \frac{2(2^n - (-1)^n)}{3}. \end{aligned}$$

Since $a_1 = 2$ and $a_2 = 6$, by solving the above linear recurrence equation we find that

$$E(X_n) = \frac{a_n}{2^n} = \frac{(3n+7) + 2(-1/2)^n}{9} \qquad \text{for any } n \geq 1.$$

$\square$