Problem 11616
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Proposed by Stefano Siboni (Italy).

Let $x_1, \ldots, x_n$ be distinct points in $\mathbb{R}^3$, and let $k_1, \ldots, k_n$ be positive real numbers. A test object at x is attracted to each of $x_1, \ldots, x_n$ with a force along the line from x to x_j of magnitude $k_j\|x - x_j\|^2$ 2, where $\|u\|$ denotes the usual euclidean norm of u. Show that when $n \geq 2$ there is a unique point x^* at which the net force on the test object is zero.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Consider the differentiable scalar function

$$f(x) = \frac{1}{3}\sum_{j=1}^{n} k_j\|x - x_j\|^3 : \mathbb{R}^3 \to \mathbb{R}.$$

Since

$$\nabla f(x) = \sum_{j=1}^{n} k_j\|x - x_j\|(x - x_j) = -\sum_{j=1}^{n} k_j\|x - x_j\|^2 u_j$$

where u_j is the unit vector along the line from x to x_j, it follows that $\nabla f(x) = 0$ is equivalent to say that the net force on a test object at x is zero.
Now, let $R \geq \max\{\|x_j\| : 1 \leq j \leq n\}$ and let $\|x\| \geq 2R$ then

$$f(x) \geq \frac{1}{3}\sum_{j=1}^{n} k_j(\|x\| - \|x_j\|)^3 \geq \frac{R^3}{3}\sum_{j=1}^{n} k_j.$$

Hence f has a local minimum in $\mathbb{R}^3$. Since f is strictly convex, we have that f has a unique global minimum at some x^* and x^* is the unique point where the gradient of f is zero. $\square$