Problem 11612
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Proposed by Paul Bracken (USA).

Evaluate in closed form

$$\prod_{n=1}^{\infty}\left(\frac{n+z+1}{n+z}\right)^n e^{(2z-2n+1)/(2n)}.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Let $z \in \mathbb{C} \setminus \{-1,-2,-3,\dots\}$. Since

$$\Gamma(z+1) \sim \frac{N!\exp(z\log N)}{\prod_{n=1}^{N}(n+z)} \sim \frac{\sqrt{2\pi}\exp(N\log N+(z+1/2)\log N-N)}{\prod_{n=1}^{N}(n+z)},$$

it follows that

$$\begin{aligned}\prod_{n=1}^{N}\left(\frac{n+z+1}{n+z}\right)^n &= \frac{N^N\left(1+\frac{1+z}{N}\right)^N}{\prod_{n=1}^{N}(n+z)}\\ &\sim \frac{\Gamma(z+1)\exp(N\log N+1+z-(N\log N+(z+1/2)\log N-N))}{\sqrt{2\pi}}\\ &\sim \frac{\Gamma(z+1)\exp(z+1-(z+1/2)\log N+N))}{\sqrt{2\pi}}.\end{aligned}$$

Hence

$$\begin{aligned}\prod_{n=1}^{N}\left(\frac{n+z+1}{n+z}\right)^n e^{(2z-2n+1)/(2n)} &\sim \frac{\Gamma(z+1)\exp(z+1-(z+1/2)\log N+N+(z+1/2)H_N-N)}{\sqrt{2\pi}}\\ &\sim \frac{\Gamma(z+1)\exp(z+1-(z+1/2)\log N+N+(z+1/2)(\log N+\gamma)-N)}{\sqrt{2\pi}}\\ &\sim \frac{\Gamma(z+1)\exp(z+1+\gamma(z+1/2))}{\sqrt{2\pi}}\end{aligned}$$

where we used the fact that $H_N=\sum_{n=1}^{N}1/n \sim \log N+\gamma$. □