Problem 11610
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Proposed by Richard P. Stanley (USA).

Let $f(n)$ be the number of binary words $a_1 \cdots a_n$ of length n that have the same number of pairs $a_i a_{i+1}$ equal to 00 *as equal to* 01*. Show that*

$$F(t) := \sum_{n=0}^{\infty} f(n)t^n = \frac{1}{2}\left(\frac{1}{1-t} + \frac{1+2t}{\sqrt{(1-t)(1-2t)(1+t+2t^2)}}\right).$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Let $p(n,a,b)$ and $q(n,a,b)$ be the number of binary words of length $n \geq 1$ that have $a \geq 0$ copies of 00, $b \geq 0$ copies of 01 and where the last bit is respectively 0 or 1. Therefore

$$P(z,x,y) := \sum_{n\geq 1, a\geq 0, b\geq 0} p(n,a,b) z^n x^a y^b = z + (1+x)z^2 + (1+x+y+x^2)z^3 + \dots,$$
$$Q(z,x,y) := \sum_{n\geq 1, a\geq 0, b\geq 0} q(n,a,b) z^n x^a y^b = z + (1+y)z^2 + (1+2y+xy)z^3 + \dots.$$

Since

$$\begin{cases} p(n+1,a,b) = xp(n,a-1,b) + q(n,a,b) & \text{for } a \geq 1 \\ q(n+1,a,b) = yp(n,a,b-1) + q(n,a,b) & \text{for } b \geq 1 \end{cases},$$

it follows that

$$\begin{cases} P(z,x,y) - P(z,0,y) = zxP(z,x,y) + z(Q(z,x,y) - Q(z,0,y)) \\ Q(z,x,y) - Q(z,0,y) = zyP(z,x,y) + z(Q(z,x,y) - Q(z,x,0)) \end{cases}$$

By using the above recurreces, it is easy to verify that

$$P(z,0,y) = \frac{z}{1-z-yz^2}, \qquad Q(z,0,y) = \frac{z(1+yz)}{1-z-yz^2}$$

and

$$P(z,x,0) = \frac{z}{(1-z)(1-xz)}, \qquad Q(z,x,0) = \frac{z}{1-z}.$$

Hence, by letting $H(z,x,y) := 1 + P(z,x,y) + Q(z,x,y)$ we obtain

$$H(z,x,y) = \frac{1+(1-x)z}{1-(1+x)z+(x-y)z^2} = 1 + 2z + (2+x+y)z^2 + (2+x+3y+xy+x^2)z^3 + \dots.$$

Now the coefficient $[z^n x^a y^b]H(z,x,y)$ is the number of the binary words of length $n \geq 0$ that have $a \geq 0$ copies of 00 and $b \geq 0$ copies of 01. Let

$$G(z,x) := H(xz, x, 1/x) = \frac{1+x(1-x)z}{1-x(1+x)z+x(x^2-1)z^2}$$

then $[z^n x^n]G(z,x) = f(n)$. In order to extract the *diagonal terms* we consider

$$G(rw, r/w) = \frac{w(r^2+1)-r^3}{r^5-r^3+(1-r^2)w-r^3w^2} = \frac{r^3-w(r^2+1)}{r^3(w-w_+)(w-w_-)}$$

with $r > 0$ sufficiently small, $w = e^{i\theta}$ and

$$w_{\pm} = \frac{r^2-1 \pm \sqrt{(1-r^2)(1-2r^2)(1+r^2+2r^4)}}{2r^3}.$$

Note that $|w_-| < 1$ and $|w_+| > 1$. Thus

$$\begin{aligned} F(r^2) &= \frac{1}{2\pi} \int_0^{2\pi} G(rw, r/w)\, d\theta = \frac{1}{2\pi i} \int_{|w|=1} \frac{G(rw, r/w)}{w}\, dw \\ &= \operatorname{Res}\left(\frac{G(rw, r/w)}{w}, w = 0 \right) + \operatorname{Res}\left(\frac{G(rw, r/w)}{w}, w = w_- \right) \\ &= \frac{1}{1 - r^2} + \frac{r^3 - w_-(r^2 + 1)}{r^3(w_- - w_+)} = \frac{1}{2} \left(\frac{1}{1 - r^2} + \frac{1 + 2r^2}{\sqrt{(1 - r^2)(1 - 2r^2)(1 + r^2 + 2r^4)}} \right) \end{aligned}$$

and the proof is complete. □