Problem 11604
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Proposed by Pál Péter Dályay (Hungary).

Given $0 \leq a \leq 2$, *let* $\{a_n\}_{n\geq 1}$ *be the sequence defined by* $a_1 = a$, $a_{n+1} = 2^n - \sqrt{2^n(2^n - a_n)}$ *for* $n \geq 1$. *Find*

$$\sum_{n=1}^{\infty} a_n^2.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Let

$$\alpha = 4\arcsin(\sqrt{a/2}) = \begin{cases} \arccos(2a^2 - 4a + 1) & \text{if } a \in [0,1], \\ 2\pi - \arccos(2a^2 - 4a + 1) & \text{if } a \in [1,2]. \end{cases}.$$

then, by using the fact that $2\cos^2(\theta/2) = 1 + \cos(\theta)$, it is easy to verify that

$$a_n = 2^{n-1}(1 - \cos(\alpha/2^n)).$$

Hence for $N > 0$

$$\begin{aligned}
\sum_{n=1}^{N} a_n^2 &= \sum_{n=1}^{N} 4^{n-1}(1 + \cos^2(\alpha/2^n) - 2\cos(\alpha/2^n)) \\
&= \sum_{n=1}^{N} 4^{n-1}\left(1 + \frac{1 + \cos(\alpha/2^{n-1})}{2} - 2\cos(\alpha/2^n)\right) \\
&= \frac{1}{2}\sum_{n=1}^{N} 4^n\left(1 - \cos(\alpha/2^n)\right) - \frac{1}{2}\sum_{n=1}^{N} 4^{n-1}\left(1 - \cos(\alpha/2^{n-1})\right) \\
&= \frac{1}{2}\sum_{n=1}^{N} 4^n\left(1 - \cos(\alpha/2^n)\right) - \frac{1}{2}\sum_{n=0}^{N-1} 4^n\left(1 - \cos(\alpha/2^n)\right) \\
&= \frac{1}{2}\left(4^N(1 - \cos(\alpha/2^N) - (1 - \cos(\alpha)\right).
\end{aligned}$$

Finally

$$\begin{aligned}
\sum_{n=1}^{\infty} a_n^2 &= \frac{1}{2}\left(\lim_{N\to+\infty} 4^N(1 - \cos(\alpha/2^N) - (1 - \cos(\alpha))\right) \\
&= \frac{\alpha^2}{4} + a^2 - 2a = 4\arcsin^2(\sqrt{a/2}) + a^2 - 2a.
\end{aligned}$$

□