Problem 11602
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Proposed by Roberto Tauraso (Italy).

Let p be a prime. Let F_n denote the nth Fibonacci number. Show that

$$\sum_{0<i<j<k<p} \frac{F_i}{ijk} \equiv 0 \pmod{p}.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Let $p>3$ be prime (if $p=2,3$ the above sum is empty and it gives 0). For $k=1,\ldots,p-1$

$$\binom{p}{k} = (-1)^{k-1}\frac{p}{k}\prod_{j=1}^{k-1}\left(1-\frac{p}{j}\right) \equiv (-1)^{k-1}\frac{p}{k}\left(1-pH_{k-1}(1)+p^2H_{k-1}(1,1)\right) \pmod{p^4}$$

$$\binom{2p}{k} = (-1)^{k-1}\frac{2p}{k}\prod_{j=1}^{k-1}\left(1-\frac{2p}{j}\right) \equiv (-1)^{k-1}\frac{2p}{k}\left(1-2pH_{k-1}(1)+4p^2H_{k-1}(1,1)\right) \pmod{p^4}$$

$$\binom{p+k-1}{k} = \frac{p}{k}\prod_{j=1}^{k-1}\left(1+\frac{p}{j}\right) \equiv \frac{p}{k}\left(1+pH_{k-1}(1)+p^2H_{k-1}(1,1)\right) \pmod{p^4}$$

where

$$H_{k-1}(1) = \sum_{0<i<k}\frac{1}{i} \quad , \quad H_{k-1}(1,1) = \sum_{0<i<j<k}\frac{1}{ij}.$$

Let us consider the following known identities: for any integer $n \geq 0$

$$(1)\ \sum_{k=0}^{n}\binom{n}{k}F_{n-k}(-1)^{n-k} = -F_n\,, \quad (2)\ \sum_{k=0}^{n}\binom{2n}{k}F_{n-k}(-1)^{n-k} = -\sum_{k=0}^{n}\binom{n+k-1}{k}F_{n-k}.$$

Letting $n=p$, then (1) and (2) become (note that $F_0=0$ and p is odd)

$$(1)\ pS_1-p^2S_2+p^3S_3 \equiv 0 \pmod{p^4}\,, \quad (2)\ 2pS_1-4p^2S_2+8p^3S_3 \equiv -pS_1-p^2S_2-p^3S_3 \pmod{p^4}$$

where

$$S_1 = \sum_{0<i<p}\frac{F_{p-i}}{i}\,, \quad S_2 = \sum_{0<i<j<p}\frac{F_{p-j}}{ij}\,, \quad S_3 = \sum_{0<i<j<k<p}\frac{F_{p-k}}{ijk}.$$

By subtracting 3 times (1) from (2) we obtain $6p^3S_3 \equiv 0 \pmod{p^4}$, that is $S_3 \equiv 0 \pmod{p}$. Finally

$$0 \equiv S_3 = \sum_{0<i<j<k<p}\frac{F_{p-k}}{ijk} = \sum_{0<i<j<k<p}\frac{F_i}{(p-k)(p-j)(p-i)} \equiv -\sum_{0<i<j<k<p}\frac{F_i}{ijk} \pmod{p}.$$

□