Problem 11601
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Proposed by Harm Derksen and Jeffrey Lagarias (USA).

The Farey series of order n is the set of reduced rational fractions j/k in the unit interval with denominator at most n. Let F_n be the product of these fractions, excluding $0/1$. That is,

$$F_n = \prod_{k=1}^{n} \prod_{\substack{j=1 \\ (j,k)=1}}^{k-1} \frac{j}{k}.$$

Let $\overline{F}_n = 1/F_n$. Show that $\overline{F}_n$ is an integer for only finitely many n.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

It follows from the Prime Number Theorem that for any real numbers $0 < \alpha < \beta$,

$$\lim_{x\to\infty} \frac{\pi(\beta x) - \pi(\alpha x)}{x/\log x} = \lim_{x\to\infty} \frac{\frac{\beta x}{\log\beta + \log x} - \frac{\alpha x}{\log\alpha + \log x}}{x/\log x} = \beta - \alpha > 0.$$

Hence

$$\lim_{x\to\infty} (\pi(\beta x) - \pi(\alpha x)) = +\infty$$

which means, in particular, that there is $n_0 > 1$ such that the open interval $(\alpha n, \beta n)$ contains at least a prime for all $n > n_0$.
We take $\alpha = 1/3$ and $\beta = 3/8$, thus for $n > n_0$ there is a prime $p \in (n/3, 3n/8)$. For any rational number $x > 0$, we denote by $v_p(x)$ the unique integer such that a $x = p^{v_p(x)} a/b$, where neither of the integers a and b is divisible by p. It is clear that if $v_p(x) < 0$ then x is not an integer because its denominator is divisible by p.

Now we determine $v_p(\overline{F}_n)$. Since $2 < p < 2p < n < 3n$, we have that

$$\begin{aligned} v_p(\overline{F}_n) &= v_p\left(\prod_{k=1}^{n} k^{\varphi(k)} \prod_{\substack{j=1 \\ (j,k)=1}}^{k-1} \frac{1}{j} \right) \\ &= v_p\left(p^{p-1} \cdot \left(\prod_{k=p+1}^{2p-1} \frac{1}{p} \right) \cdot (2p)^{p-1} \cdot \left(\prod_{k=2p+1}^{n} \frac{1}{p} \right) \cdot \left(\prod_{\substack{k=2p+1 \\ (2,k)=1}}^{n} \frac{1}{2p} \right) \right) \\ &= p - 1 - (2p-1-p) + p - 1 - (n-2p) - (\lceil n/2 \rceil - p) = -\lceil 3n/2 + 1 - 4p \rceil \end{aligned}$$

which is negative because $4p < 4(3n/8) = 3n/2$. Therefore $\overline{F}_n$ is not integer for any $n > n_0$. $\square$