Problem 11597
(American Mathematical Monthly, Vol.118, October 2011)

Proposed by Michel Bataille (France).

Let $f(x) = x/\log(1-x)$. *Prove that for* $0 < x < 1$,

$$\sum_{n=1}^{\infty} \frac{x^n(1-x)^n}{n!}\, f^{(n)}(x) = -\frac{1}{2}\, x f(x).$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

The function f has a power expansion at 0 given by $\sum_{k=1}^{\infty} a_k x^k$ for $x \in (0,1)$. Moreover, for $m \geq 1$,

$$\begin{aligned}
[x^m]\sum_{n=1}^{\infty} \frac{x^n(1-x)^n}{n!}\, f^{(n)}(x) &= [x^m]\sum_{n=1}^{\infty} x^n \sum_{j=0}^{n}\binom{n}{j}\sum_{k=n}^{\infty}\binom{k}{n} a_k x^{k-n} \\
&= \sum_{n=1}^{m}\sum_{k=n}^{m}\binom{n}{m-k}(-1)^{m-k}\binom{k}{n} a_k \\
&= \sum_{k=1}^{m}(-1)^{m-k} a_k \sum_{n=1}^{k}\binom{n}{m-k}\binom{k}{n} \\
&= \sum_{k=1}^{m}(-1)^{m-k} a_k \left(2^{2km}\binom{k}{m-k} - \binom{0}{m-k}\right).
\end{aligned}$$

Hence, the identity holds as soon as

$$\sum_{k=1}^{m}(-1)^{m-k} a_k \left(2^{2km}\binom{k}{m-k} - \binom{0}{m-k}\right) = [x^m] - \frac{1}{2}\, x f(x) = -\frac{1}{2}\, a_{m-1},$$

that is if and only if

$$\sum_{k=1}^{m}\binom{k}{m-k}(-4)^k a_k = (-2)^m a_m + (-2)^{m-1} a_{m-1}.$$

Since $\sum_{k=1}^{m}\binom{k}{m-k}[t^k]g(t) = [t^m]g(t(1+t))$, it follows that

$$\begin{aligned}
\sum_{k=1}^{m}\binom{k}{m-k}(-4)^k a_k &= [t^m]f(-4t) = [t^m]\frac{-4t(1+t)}{\log(1+4t+t^2)} \\
&= [t^m]f(-2t) + [t^{m-1}]f(-2t) = (-2)^m a_m + (-2)^{m-1} a_{m-1}
\end{aligned}$$

and the proof is complete. $\square$