Problem 11594
(American Mathematical Monthly, Vol.118, October 2011)

Proposed by Harm Derksen and Jeffrey Lagarias (USA).

Let

$$G_n = \prod_{k=1}^{n} \left(\prod_{j=1}^{k-1} \frac{j}{k} \right),$$

and let $\overline{G}_n = 1/G_n$.
(a) *Show that if* n *is an integer greater than* 1, *then* $\overline{G}_n$ *is an integer.*
(b) *Show that for each prime* p, *there are infinitely many* n *greater than* 1 *such that* p *does not divide* $\overline{G}_n$.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

The number $\overline{G}_n$ is an integer because

$$\overline{G}_n = \prod_{k=1}^{n} \left(\prod_{j=1}^{k-1} \frac{k}{j} \right) = \prod_{k=1}^{n} \frac{k^k}{k!} = \prod_{k=1}^{n-1} \binom{n}{k}.$$

Moreover, if p is a prime then by Lucas' Theorem

$$\binom{n}{k} \equiv \binom{n_{s-1}}{k_{s-1}} \binom{n_{s-2}}{k_{s-2}} \cdots \binom{n_0}{k_0} \pmod{p}$$

where $n = n_{s-1}p^{s-1} + n_{s-2}p^{s-2} + \cdots + n_0$ and $k = k_{s-1}p^{s-1} + k_{s-2}p^{s-2} + \cdots + k_0$ are the base p expansions of n and k respectively. Hence, by letting $n = p^s - 1$ with $s > 0$ then for $k = 1, \ldots, n-1$

$$\binom{n}{k} \equiv \binom{p-1}{k_{s-1}} \binom{p-1}{k_{s-2}} \cdots \binom{p-1}{k_0} \equiv (-1)^{k_{s-1}} \cdot (-1)^{k_{s-2}} \cdots (-1)^{k_0} \not\equiv 0 \pmod{p}.$$

Therefore p does not divide $\overline{G}_n = \prod_{k=1}^{n-1} \binom{n}{k}$. □