Problem 11590
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Let m balls numbered 1 to m each be painted with one of n colors, with $n \geq 2$ and at least two balls of each color. For each positive integer k, let $P(k)$ be the number of ways to put these balls into urns numbered 1 through k so that no urn is empty and no urn gets two or more balls of the same color. Prove that

$$\sum_{k=1}^{m} \frac{(-1)^k}{k} P(k) = 0.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

We assume that among the m balls numbered 1 to m there are $x_i \geq 1$ balls of the i-th color for $i = 1, \ldots, n$ with $\sum_{i=1}^{n} x_i = m$. Let us denote by $P([x_1, \ldots, x_n], k)$ the number of ways to put these balls into urns numbered 1 through k so that no urn is empty and no urn gets two or more balls of the same color. We will show that the desired equality holds by induction with respect to the number of colors n.

It is easy to verify that for $n = 2$ we have that

$$P([x_1, x_2], k) = \binom{x_1}{x_1 + x_2 - k}\binom{x_2}{x_1 + x_2 - k}(x_1 + x_2 - k)!k! = \binom{x_1}{x_1 + x_2 - k}\binom{k}{x_1} x_1!x_2!,$$

and for $n > 2$ the following recurrence relation holds

$$P([x_1, \ldots, x_n], k) = \sum_{j=1}^{k} \frac{P([x_1, \ldots, x_{n-1}], j)}{j!} \cdot P([x_n, j], k).$$

Note that $P([x_1, \ldots, x_n], k) = 0$ if $x_i > k$ for some i or if $k > m$.

We first prove the equality for $n = 2$:

$$\begin{aligned}
\sum_{k=1}^{m} \frac{(-1)^k}{k} P([x_1, x_2], k) &= (x_1 - 1)!x_2! \sum_{k=1}^{m} (-1)^k \binom{x_1}{m - k}\binom{k - 1}{x_1 - 1} \\
&= (x_1 - 1)!x_2!(-1)^m \sum_{i=0}^{x_1} (-1)^i \binom{x_1}{i}\binom{m - 1 - i}{x_1 - 1} \\
&= (x_1 - 1)!x_2!(-1)^m [t^{m-1}]\, (1 - t)^{x_1} \cdot \frac{t^{x_1 - 1}}{(1 - t)^{x_1}} \\
&= (x_1 - 1)!x_2!(-1)^m [t^{m - x_1}]\, 1 = 0.
\end{aligned}$$

Finally, for $n > 2$, by the previous identity and the recurrence relation we get

$$\begin{aligned}
\sum_{k=1}^{m} \frac{(-1)^k}{k} P([x_1, \ldots, x_n], k) &= \sum_{k=1}^{m} \frac{(-1)^k}{k} \sum_{j=1}^{k} \frac{P([x_1, \ldots, x_{n-1}], j)}{j!} \cdot P([j, x_n], k) \\
&= \sum_{j=1}^{m} \frac{P([x_1, \ldots, x_{n-1}], j)}{j!} \cdot \sum_{k=j}^{m} \frac{(-1)^k}{k} P([j, x_n], k) \\
&= \sum_{j=1}^{\sum_{i=1}^{n-1} x_i} \frac{P([x_1, \ldots, x_{n-1}], j)}{j!} \cdot \sum_{k=1}^{j + x_n} \frac{(-1)^k}{k} P([j, x_n], k) = 0
\end{aligned}$$

and the proof is complete.

Remark. The desired equality holds without the assumption that there are at least two balls of each color. Moreover, if $x_1 = \cdots x_n = 1$ then $P([x_1, \ldots, x_n], k)/k!$ is equal to the Stirling number of the second kind $\left\{ n \atop k \right\}$ and the equality is a known property of this sequence:

$$\sum_{k=1}^{n} (-1)^k (k-1)! \left\{ n \atop k \right\} = 0.$$

□