Problem 11583
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Proposed by David Beckwith (USA).

The instructions for a magic trick are as follows. Pick a positive integer n. Next, list all partitions of n as nondecreasing strings. For instance, with $n = 3$, the list is $\{111, 12, 3\}$. Count 1 point for the string (n). For the string $\lambda_1 \dots \lambda_k$ with $k > 1$, count $\prod_{j=1}^{k-1} \binom{\lambda_{j+1}}{\lambda_j}$ points. Add up your points, take the log *base 2 of that, and add 1. Voilà! n. Explain.*

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Let $\lambda_1 \leq \cdots \leq \lambda_k$ be a partition of n, then the conjugate partition is given by λ_1 copies of k, $\lambda_2 - \lambda_1$ copies of $k-1$,..., and $\lambda_k - \lambda_{k-1}$ copies of 1.
By taking all the permutations of these $m = \lambda_k$ positive integers we obtain

$$\binom{\lambda_k}{\lambda_1, \lambda_2 - \lambda_1, \dots, \lambda_k - \lambda_{k-1}}$$

solutions of the diophantine linear equation

$$x_1 + x_2 + \cdots + x_m = n \quad \text{with } x_i \geq 1 \text{ for } i = 1, \dots, m.$$

It is easy to see that this correspondence is bijective.
Hence, let $\mathcal{P}_n$ be the set of all partitions $\lambda_1 \leq \cdots \leq \lambda_k$ of n, then

$$\begin{aligned}
\sum_{\mathcal{P}_n} \prod_{j=1}^{k-1} \binom{\lambda_{j+1}}{\lambda_j} &= \sum_{\mathcal{P}_n} \binom{\lambda_k}{\lambda_1, \lambda_2 - \lambda_1, \dots, \lambda_k - \lambda_{k-1}} \\
&= \sum_{m=1}^{n} |(x_1, x_2, \dots, x_m) \in \mathbb{N} : x_i \geq 1,\ x_1 + x_2 + \cdots + x_m = n| \\
&= \sum_{m=1}^{n} \binom{n - m + (m-1)}{m-1} = \sum_{m=1}^{n} \binom{n-1}{m-1} = 2^{n-1}
\end{aligned}$$

and the proof is complete. □