Problem 11577
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Let n be a positive even integer and let p be prime. Show that the polynomial f given by $f(z) = p + \sum_{k=1}^{n} z^k$ is irreducible over $\mathbb{Q}$.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

We prove a more general statement by assuming that n is a positive integer (not necessarly even). By Gauss' Lemma, f is irreducible over $\mathbb{Q}$ if and only if it is irreducible over the $\mathbb{Z}$. Assume by contradiction that $f(z) = g(z)h(z)$, where g and h are monic polynomials of positive degree with integer coefficients. The product of the constant terms of g and h is equal to p and, since p is prime, one of these constant terms is equal to ± 1.

Therefore the product of the absolute values of the roots of one of the polynomials g and h is equal to 1. Hence, this polynomial must have a root $z_0 \in D := \{z \in \mathbb{C} : |z| \leq 1\}$. Since z_0 is also a root of f, we have that

$$0 = f(z_0) \cdot (z_0 - 1) = z_0^{n+1} + (p-1)z_0 - p.$$

Therefore

$$\frac{z_0^{n+1} + (p-1)z_0}{p} = 1.$$

which means that the point 1 is a proper convex combination of points $z_0^{n+1}, z_0 \in D$. Since D is strictly convex and $1 \in \partial D$, it follows that $z_0^{n+1} = z_0 = 1$. On the other hand 1 is not a root of f because $f(1) = n + p > 0$. $\square$