Problem 11570
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Proposed by Kirk Bresniker and Stan Wagon (USA).

Alice and Bob play a number game. Starting with a positive integer n, they take turns changing the number; Alice goes first. Each player in turn may change the current number k to either $k-1$ or $\lceil k/2 \rceil$. The person who changes 1 to 0 wins. For instance, when $n = 3$, the players have no choice, k proceeds from 3 to 2 to 1 to 0, and Alice wins. When $n = 4$, Alice wins if and only if her first move is to 2. For which initial n does Alice have a winning strategy?

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

The set of the winning positions for Alice is

$$\begin{aligned} W &= \{1\} \cup \{2^k + 1 : k \geq 1 \text{ odd}\} \cup \{2^k d + 1 : k \geq 0 \text{ even and } d \geq 3 \text{ odd}\} \\ &= \{1, 3, 4, 6, 8, 9, 10, 12, 13, 14, 16, 18, 20, 21, 22, 24, 26, 28, 29, 30, 32, 33, 34, 36, 37, 38, 40, \dots\}. \end{aligned}$$

Let $L := \mathbb{N} \setminus A$. It is easy to verify that if the current number is $k \in W$ then either $k - 1 \in L$ or $\lceil k/2 \rceil \in L$ (when the condition holds then that is the right move for Alice).
On the other hand, if the current number is $k \in L$ then both $k - 1 \in W$ and $\lceil k/2 \rceil \in W$. $\square$