Problem 11568
(American Mathematical Monthly, Vol.118, April 2011)

Proposed by Kurt Foster (USA).

For $n \geq 1$, let $f(n)$ be the least-significant nonzero decimal digit of $n!$.
For $n \geq 2$, show that $f(625n) = f(n)$.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

We first note that if $2^{d_1}3^{d_2}5^{d_3}\cdots p_r^{d_r}$ is the prime factorization of $n!$ then $d_1 \geq d_3$ and

$$f(n) = [2^{d_1-d_3}3^{d_2}\cdots p_r^{d_r}]_{\pmod{10}}.$$

Since $d_1 > d_3$ for $n \geq 2$, it follows that $f(n) \in \{2,4,6,8\}$. Hence, by the Chinese Remainder Theorem, it suffices to prove that

$$f(625n) \equiv f(n) \pmod 5.$$

Let $n \geq 1$, then

$$(5n)! = 5^n n! \cdot \prod_{k=0}^{n-1}(5k+1)(5k+2)(5k+3)(5k+4) = 10^n n! \cdot 12^n \prod_{k=0}^{n-1}\binom{5k+4}{4}$$

and $f(5n)$ is the least-significant nonzero decimal digit of

$$n! \cdot 12^n \prod_{k=0}^{n-1}\binom{5k+4}{4}.$$

Moreover, by Lucas' Theorem $\binom{5k+4}{4} \equiv 1 \pmod 5$, and it follows that

$$f(5n) \equiv 2^n f(n) \pmod 5.$$

Finally, by Fermat's Theorem $2^{156} = (2^4)^{39} \equiv 1 \pmod 5$, and by applying four times the previous congruence, we obtain

$$f(625n) \equiv 2^{(125+25+5+1)n} f(n) \equiv 2^{(156)n} f(n) \equiv f(n) \pmod 5.$$

□