Problem 11564
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Proposed by Albert Stadler (Switzerland).

Prove that

$$\int_0^\infty \frac{e^{-x}(1-e^{-6x})}{x(1+e^{-2x}+e^{-4x}+e^{-6x}+e^{-8x})}\,dx = \log\left(\frac{3+\sqrt{5}}{2}\right).$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

We will prove a more general result: if $a, b, c > 0$ and $d = a + b + 2c$ then

$$\int_0^\infty \frac{(1-e^{-ax})(1-e^{-bx})e^{-cx}}{x(1-e^{-dx})}\,dx = \log\left(\frac{\sin(\pi(a+c)/d)}{\sin(\pi c/d)}\right).$$

By letting $a = 2$, $b = 6$, and $c = 1$, then the integral is equal to

$$\log\left(\frac{\sin(3\pi/10)}{\sin(\pi/10)}\right) = \log\left(\frac{3+\sqrt{5}}{2}\right).$$

By expanding the integrand, we obtain

$$\begin{aligned}
\int_0^\infty \frac{(1-e^{-ax})(1-e^{-bx})e^{-cx}}{x(1-e^{-dx})}\,dx &= \int_0^\infty \sum_{n\geq 1}\frac{(-1)^n((a+b)^n-a^n-b^n)x^{n-1}}{n!}\sum_{k\geq 0} e^{-(kd+c)x}\,dx\\
&= \sum_{n\geq 1}\frac{(-1)^n((a+b)^n-a^n-b^n)}{n!}\sum_{k\geq 0}\int_0^\infty x^{n-1}e^{-(kd+c)x}\,dx\\
&= \sum_{k\geq 0}\sum_{n\geq 1}\frac{(-1)^n((a+b)^n-a^n-b^n)}{n(kd+c)^n}\\
&= \sum_{k\geq 0}\left(\log\left(1+\frac{a}{kd+c}\right)+\log\left(1+\frac{b}{kd+c}\right)-\log\left(1+\frac{a+b}{kd+c}\right)\right)\\
&= \sum_{k\geq 0}\log\left(\frac{(kd+a+c)(kd+b+c)}{(kd+c)(kd+a+b+c)}\right)\\
&= \log\prod_{k\geq 0}\left(\frac{(kd+a+c)(kd+b+c)}{(kd+c)(kd+a+b+c)}\right)\\
&= \log\left(\frac{\Gamma(c/d)\Gamma((a+b+c)/d)}{\Gamma((a+c)/d)\Gamma((b+c)/d)}\right)
\end{aligned}$$

where in the last step we used the Weirstarss product

$$z\prod_{k\geq 1}\left(1+\frac{z}{k}\right)e^{-\frac{z}{k}} = \frac{1}{\Gamma(z)e^{\gamma z}}.$$

By the reflection property, if z is not an integer then

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}.$$

Hence, since $d = a + b + 2c$, it follows that

$$\frac{\Gamma(c/d)\Gamma((a+b+c)/d)}{\Gamma((a+c)/d)\Gamma((b+c)/d)} = \frac{\sin(\pi(a+c)/d)}{\sin(\pi c/d)}.$$

□