Problem 11559
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Proposed by Michel Bataille (France).

For positive p and $x \in (0,1)$, define the sequence $\{x_n\}_{n\geq 0}$ by $x_0 = 1$, $x_1 = x$, and, for $n \geq 1$,

$$x_{n+1} = \frac{p x_{n-1} x_n + (1-p) x_n^2}{(1+p)x_{n-1} - p x_n}.$$

Find positive real numbers α, β such that $\lim_{n\to\infty} n^{\alpha} x_n = \beta$.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

We will prove that

$$\alpha = \frac{1}{p} \qquad \text{and} \qquad \beta = \frac{\Gamma\left(\frac{1}{p(1-x)}\right)}{\Gamma\left(\frac{x}{p(1-x)}\right)}.$$

By letting $a_{n+1} = x_{n+1}/x_n$, it is easy to see that

$$a_{n+1} = \frac{p + (1-p)a_n}{(1+p) - p a_n}$$

which implies that

$$\frac{1}{1 - a_{n+1}} = \frac{1}{1-a_n} + p = \frac{1}{1-x} + pn.$$

Hence

$$x_n = \prod_{k=0}^{n-1} \left(1 - \frac{z}{c+k}\right)$$

where $z = 1/p > 0$ and $c = z/(1-x) > 0$. We note that

$$\lim_{n\to\infty} n^z \prod_{k=0}^{n-1} e^{-\frac{z}{c+k}} = \lim_{n\to\infty} e^{z\left(\log n - \sum_{k=0}^{n-1} \frac{1}{c+k}\right)} = e^{z\Psi(c)}$$

where Ψ is the Digamma function (remember that $\Psi(x+1) - \Psi(x) = 1/x$ and $\Psi(x) = \log x + O(1/x)$). Finally, by using the Mellin's Formula, we obtain

$$\lim_{n\to\infty} n^{\alpha} x_n = \lim_{n\to\infty} n^z x_n = e^{z\Psi(c)} \lim_{n\to\infty} \prod_{k=0}^{n-1} \left(1 - \frac{z}{c+k}\right) e^{\frac{z}{c+k}} = \frac{\Gamma(c)}{\Gamma(c-z)} = \beta.$$

□