Problem 11555
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Let f be a continuous real-valued function on $[0,1]$ such that $\int_0^1 f(x)\,dx = 0$. Prove that there exists c in the interval $(0,1)$ such that

$$c^2 f(c) = \int_0^c (x + x^2) f(x)\,dx.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Let

$$F(x) = \int_0^x (t^2 + (1-x)t) f(t)\,dt$$

then F is differentiable in $[0,1]$ and

$$F'(x) = (x^2 + (1-x)x) f(x) - \int_0^x t f(t)\,dt = x f(x) - \int_0^x t f(t)\,dt.$$

It suffices to prove that there are $a, b \in (0,1]$ such that $F'(a) > 0$ and $F'(b) < 0$.
In fact, since F' is continuous, it follows that there is d between a and b such that $F'(d) = F'(0) = 0$ and by the Flett's Mean Value Theorem there exists $c \in (0,d)$ such that $F(c) - F(0) = cF'(c)$ that is

$$\int_0^c (t^2 + (1-c)t) f(t)\,dt = c^2 f(c) - c\int_0^c t f(t)\,dt$$

and the desired equality is verified.
Since f is a continuous in $[0,1]$, it follows that there exist $x_M, x_m \in [0,1]$ such that

$$M := \max_{x\in[0,1]} f(x) = f(x_M) \qquad \text{and} \qquad m := \max_{x\in[0,1]} f(x) = f(x_m).$$

By hypothesis, $\int_0^1 f(x)\,dx = 0$, and if f is not identically zero (otherwise the thesis is trivial) then $M > 0$ and $m < 0$. Moreover, if $x_M > 0$ then let $a = x_M$ and

$$F'(a) \geq M x_M - M \int_0^{x_M} t\,dt = M x_M (1 - x_M/2) > 0.$$

On the other hand, if $x_M = 0$ then, by continuity, there is $0 < a \leq 1$ such that $f(x) \geq 3M/4$ for $x \in [0,a]$ and

$$F'(a) \geq \frac{3}{4} M a - M \int_0^a t\,dt = \frac{1}{4} M a (3 - 2a) > 0.$$

In a similar way, if $x_m > 0$ then let $b = x_m$ and

$$F'(b) \leq m x_m - m \int_0^{x_m} t\,dt = m x_m (1 - x_m/2) < 0.$$

On the other hand, if $x_m = 0$ then, by continuity, there is $0 < b \leq 1$ such that $f(x) \leq 3m/4$ for $x \in [0,b]$ and

$$F'(b) \leq \frac{3}{4} m b - m \int_0^b t\,dt = \frac{1}{4} m b (3 - 2b) < 0.$$

□