Problem 11553
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Proposed by Mihaly Bencze (Romania).

For a positive integer k, let $\alpha(k)$ be the largest odd divisor of k. Prove that for each positive integer n,

$$\frac{n(n+1)}{3} \leq \sum_{k=1}^{n} \frac{n-k+1}{k}\,\alpha(k) \leq \frac{n(n+3)}{3}.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma “Tor Vergata”, via della Ricerca Scientifica, 00133 Roma, Italy.

Let

$$S_0(n) = \sum_{k=1}^{n} \alpha(k) \qquad \text{and} \qquad S_1(n) = \sum_{k=1}^{n} \frac{\alpha(k)}{k}.$$

Since $\alpha(2k-1) = 2k-1$ and $\alpha(2k) = \alpha(k)$ it follows that for $n \geq 1$

$$S_0(n) = \sum_{k=1}^{\lfloor n/2 \rfloor} \alpha(2k) + \sum_{k=1}^{\lceil n/2 \rceil} \alpha(2k-1) = S_0(\lfloor n/2 \rfloor) + \lceil n/2 \rceil^2,$$

and in a similar way we have that

$$S_1(n) = \sum_{k=1}^{\lfloor n/2 \rfloor} \frac{\alpha(2k)}{2k} + \sum_{k=1}^{\lceil n/2 \rceil} \frac{\alpha(2k-1)}{2k-1} = \frac{1}{2} S_1(\lfloor n/2 \rfloor) + \lceil n/2 \rceil.$$

Therefore

$$\begin{aligned} T(n) := \sum_{k=1}^{n} \frac{n-k+1}{k}\,\alpha(k) &= (n+1)S_1(n) - S_0(n) \\ &= T(\lfloor n/2 \rfloor) + \lceil n/2 \rceil(\lfloor n/2 \rfloor + 1) - \frac{[2 \mid n]}{2}\, S_1(\lfloor n/2 \rfloor) \end{aligned}$$

where $[2 \mid n]$ is equal to 1 if n is even and it is 0 otherwise.
By using the recurrence for $S_1(n)$, it is easy to verify by induction that

$$S_1(n) \leq \frac{2\lceil n/2 \rceil + n + 1}{3}.$$

Hence, since $0 \leq S_1(\lfloor n/2 \rfloor) = 2(S_1(n) - \lceil n/2 \rceil)$ and $n = \lfloor n/2 \rfloor + \lceil n/2 \rceil$,

$$T(\lfloor n/2 \rfloor) + \lceil n/2 \rceil(\lfloor n/2 \rfloor + 1) - \frac{[2 \mid n]}{3}\,(\lfloor n/2 \rfloor + 1) \leq T(n) \leq T(\lfloor n/2 \rfloor) + \lceil n/2 \rceil(\lfloor n/2 \rfloor + 1).$$

The upper bound follows by induction by verifying that

$$\lfloor n/2 \rfloor(\lfloor n/2 \rfloor + 3) + 3\lceil n/2 \rceil(\lfloor n/2 \rfloor + 1) \leq n(n+3).$$

Similarly the lower bound follows by induction by verifying that

$$n(n+1) \leq \lfloor n/2 \rfloor(\lfloor n/2 \rfloor + 1) + 3\lceil n/2 \rceil(\lfloor n/2 \rfloor + 1) - [2 \mid n]\,(\lfloor n/2 \rfloor + 1).$$

$\square$