Problem 11548
(American Mathematical Monthly, Vol.118, January 2011)

Proposed by Cezar Lupu (Romania) and Tudorel Lupu (Romania).

Let f be a twice-differentiable real-valued function with continuous second derivative, and suppose that $f(0) = 0$. Show that

$$\int_{-1}^{1} (f''(x))^2 \, dx = 10 \left(\int_{-1}^{1} f(x) \, dx \right)^2 .$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Let

$$g(x) = \begin{cases} (x+1)^2 & \text{for } x \in [-1, 0], \\ (x-1)^2 & \text{for } x \in [0, 1] \end{cases}$$

then

$$g(-1) = g(1) = g'(-1) = g'(1) = 0, \quad g(0) = 1, \quad \text{and} \quad g''(x) = 2 \text{ for } x \in [-1, -1] \setminus \{0\},$$

and

$$\int_{-1}^{1} (g(x))^2 \, dx = \int_{-1}^{0} (x+1)^4 \, dx + \int_{0}^{1} (x-1)^4 \, dx = \frac{2}{5}.$$

Therefore, since $f(0) = 0$,

$$\begin{aligned} \int_{0}^{1} g(x) f''(x) \, dx &= [g(x) f'(x)]_0^1 - \int_{0}^{1} g'(x) f'(x) \, dx \\ &= -f'(0) - [g'(x) f(x)]_0^1 + \int_{0}^{1} g''(x) f(x) \, dx = -f'(0) + 2 \int_{0}^{1} f(x) \, dx. \end{aligned}$$

In a similar way, we have that

$$\int_{-1}^{0} g(x) f''(x) \, dx = f'(0) + 2 \int_{0}^{1} f(x) \, dx.$$

Hence, by Cauchy-Schwarz inequality,

$$\frac{2}{5} \cdot \int_{-1}^{1} (f''(x))^2 \, dx = \int_{-1}^{1} (g(x))^2 \, dx \cdot \int_{-1}^{1} (f''(x))^2 \, dx \geq \left(\int_{-1}^{1} g(x) f''(x) \, dx \right)^2 = \left(2 \int_{-1}^{1} f(x) \, dx \right)^2 ,$$

and the result follows immediately. □