Problem 11546
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Let d, k, and q be positive integers, with k odd. Find the highest power of 2 that divides

$$S_q(2^d k) = \sum_{n=1}^{2^d k} n^q.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

We will prove that if $d \geq 1$ and k is odd then there exists an odd number $A_q(2^d k)$ such that

$$S_q(2^d k) = A_q(2^d k) \cdot \begin{cases} 2^{d-1} & \text{if } q = 1 \text{ or } q \text{ is even,} \\ 2^{2(d-1)} & \text{if } q \geq 3 \text{ is odd} \end{cases}.$$

Therefore

$$\text{ord}_2(S_q(2^d k)) = \begin{cases} d-1 & \text{if } q = 1 \text{ or } q \text{ is even,} \\ 2(d-1) & \text{if } q \geq 3 \text{ is odd} \end{cases}.$$

We first note that for any integer $m \geq 1$

$$\begin{aligned} S_q(2m) &= m^q + \sum_{j=1}^{n} ((m-j)^q + (m+j)^q) = m^q + \sum_{j=1}^{n} \sum_{r=0}^{q} \binom{q}{r} m^{q-r} ((-j)^r + j^r) \\ &= m^q + 2 \sum_{r=0}^{\lfloor q/2 \rfloor} \binom{q}{2r} m^{q-2r} S_{2r}(m) = m^q + 2m^{q+1} + 2 \sum_{r=1}^{\lfloor q/2 \rfloor} \binom{q}{2r} m^{q-2r} S_{2r}(m). \end{aligned}$$

Our statement is true for $q = 1$ because $S(2m) = m(2m+1)$.
Now we assume $q \geq 2$ and we proceed by induction on d. The statement is true for $d = 1$ because

$$S_q(2k) \equiv \sum_{n=1}^{2k} n = k(2k+1) \equiv 1 \pmod 2.$$

Moreover, if $q \geq 2$ is even then $qd \geq d + 1$ and

$$S_q(2^{d+1}k) = (2^d k)^q + 2(2^d k)^{q+1} + 2 \sum_{r=1}^{q/2} \binom{q}{2r} (2^d k)^{q-2r} A_{2r}(2^d k) 2^{d-1} \equiv A_q(2^d k) 2^d \pmod{2^{d+1}}$$

where $A_q(2^d k)$ is odd. On the other hand, if $q \geq 3$ is odd then $qd \geq 2d + 1$ and

$$S_q(2^{d+1}k) = (2^d k)^q + 2(2^d k)^{q+1} + 2 \sum_{r=1}^{(q-1)/2} \binom{q}{2r} (2^d k)^{q-2r} A_{2r}(2^d k) 2^{d-1} \equiv qk A_{q-1}(2^d k)\, 2^{2d} \pmod{2^{2d+1}}$$

where $qkA_{q-1}(2^d k)$ is odd. $\square$