Problem 11545
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Proposed by Manuel Kauers (Austria) and Sheng-Lan Ko(Taiwan).

Find a closed-form expression for

$$\sum_{k=0}^{n}(-1)^k\binom{2n}{n+k}s(n+k,k)$$

where s refers to the (signed) Stirling numbers of the first kind.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Let $\left[{n \atop k}\right]$ be the unsigned Stirling numbers of the first kind. We recall that $\left[{n \atop k}\right]$ counts the number of permutations of $\{1,2,\dots,n\}$ with k cycles. Since $s(n,k)=(-1)^{n-k}\left[{n \atop k}\right]$, it follows that

$$a_n := \sum_{k=0}^{n}(-1)^k\binom{2n}{n+k}s(n+k,k)=\sum_{k=0}^{n}(-1)^{n-k}\binom{2n}{n-k}\left[{n+k \atop k}\right]=\sum_{k=0}^{n}(-1)^k\binom{2n}{k}\left[{2n-k \atop n-k}\right].$$

The last sum can be seen as an application of the Inclusion-Exclusion Principle where the product

$$\binom{2n}{k}\left[{2n-k \atop n-k}\right]$$

counts the number of permutations of the set $\{1,2,\dots,2n\}$ with n cycles where at least k of them are of length 1 (we choose k numbers in $\binom{2n}{k}$ ways and then we complete the permutation in $\left[{2n-k \atop n-k}\right]$ ways). Hence a_n is the number of permutations of the set $\{1,2,\dots,2n\}$ with n cycles all of length greater than 1, that is with all cycles of length 2. Therefore

$$a_n=\binom{2n}{n}\frac{n!}{2^n}=\prod_{k=1}^{n}(2k-1)=(2n-1)!!.$$

□