**Problem 11544**
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Proposed by Max A. Alekseyev (USA) and Frank Ruskey(Canada).

*Prove that if $m$ is a positive integer, then*

$$\sum_{k=0}^{m-1} \varphi(2k+1) \left\lfloor \frac{m+k}{2k+1} \right\rfloor = m^2.$$

*Here $\varphi$ denotes the Euler totient function.*

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

We prove the statement by induction with respect to $m \geq 1$.
For $m = 1$, it is trivial. Since, for any positive integers $n$ and $k$,

$$\left\lfloor \frac{n+1}{k} \right\rfloor - \left\lfloor \frac{n}{k} \right\rfloor = \begin{cases} 1 & \text{if } k \mid (n+1), \\ 0 & \text{otherwise} \end{cases},$$

it follows that

$$\sum_{k=0}^{m} \varphi(2k+1) \left\lfloor \frac{m+1+k}{2k+1} \right\rfloor = \sum_{k=0}^{m} \varphi(2k+1) \left( \left\lfloor \frac{m+k}{2k+1} \right\rfloor + [(2k+1) \mid (m+1+k)] \right)$$

where $[Q]$ is equal to 1 when the proposition $Q$ is true and 0 otherwise.
We note that

$$[(2k+1) \mid (m+1+k)] = [(2k+1) \mid 2(m+1+k)] = [(2k+1) \mid (2m+1)].$$

Therefore

$$\sum_{k=0}^{m} \varphi(2k+1)[(2k+1) \mid (m+1+k)] = \sum_{k=0}^{m} \varphi(2k+1)[(2k+1) \mid (2m+1)] = \sum_{d \mid (2m+1)} \varphi(d) = 2m+1$$

Hence, by the inductive hypothesis,

$$\sum_{k=0}^{m} \varphi(2k+1) \left\lfloor \frac{m+1+k}{2k+1} \right\rfloor = m^2 + (2m+1) = (m+1)^2.$$

□