Problem 11533
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Proposed by E. Just (USA).

Let t be a positive integer and let R be a ring, not necessarily having an identity element, such that $x + x^{2t+1} = x^{2t} + x^{10t+1}$ (*) for each x in R. Prove that R is a Boolean ring, that is, $x = x^2$ for all x in R.

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We first note that R has no non-trivial nilpotent elements: if $x^r = 0$ with $r \geq 1$ then $x = 0$. Assume by contradiction that $x \neq 0$ and $r \geq 2$ is the minimal positive exponent such that $x^r = 0$. By (*)

$$x^{r-1} = x^{10t+r-1} + x^{2t+r-2} - x^{2t+r-1} = 0$$

against the minimality of r.
Let $I = x^{10t} - x^{2t} + x^{2t-1}$ then $Ix = x$. Moreover, it is easy to see by induction that

$$T^a x^b = x^{b-a}, \quad T^a x^a = Tx = x^{10t} - x^{2t} + x^{2t-1}, \quad \text{for } b > a > 0$$

where $T = x^{10t-1} - x^{2t-1} + Ix^{2t-2}$. Hence, by (*)

$$\begin{aligned}
0 &= (T^{10t+1} - T^{2t+1} + T^{2t} - T)x^{8t+3} \\
&= T^{2t-2}(T^{8t+3}x^{8t+3}) - x^{6t+2} + x^{6t+3} - x^{8t+2} \\
&= T^{2t-2}(x^{10t} - x^{2t} + x^{2t-1}) - x^{6t+2} + x^{6t+3} - x^{8t+2} \\
&= x^{8t+2} - x^2 + x - x^{6t+2} + x^{6t+3} - x^{8t+2} = x^{6t+1}(x^2 - x) - (x^2 - x),
\end{aligned}$$

that is, by letting $P = x^2 - x$,

$$x^{6t+1}P = P. \tag{1}$$

Let $Q = x^{2t} - x$, then by (*) and by (1)

$$-QP = x^{10t+1}P - x^{2t+1}P = x^{4t}P - x^{2t+1}P = x^{2t}QP.$$

Since Q and P commute, it follows by (1) that

$$PQ = x^{6t+1}PQ = x^{6t+1}QP = x^{4t+1}(-QP) = x^{2t+1}QP = -xQP.$$

Therefore

$$(QP)^2 = (x^{2t}QP - xQP)P = (-QP + PQ)P = 0$$

and, by the first remark, $QP = 0$, that is $x^{2t}P = xP$. Hence by (1)

$$P = x^{6t+1}P = x^{4t+2}P = x^{2t+3}P = x^4 P. \tag{2}$$

Finally, by (1) and (2)

$$P^2 = x^2P - xP = x^{2+(6t-1)\cdot(6t+1)}P - x^{1+9t^2\cdot 4}P = 0$$

which implies that $P = 0$, that is $x^2 = x$. □

Remark. Note that $2x = (2x)^2 = (x + x)^2 = 4x^2 = 4x$ and therefore $2x = 0$, which means that R has characteristic 2. Moreover, by Jacobson's Commutativity Theorem, $x^2 = x$ implies that R is commutative.