Problem 11520
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Proposed by P. Ash (USA).

Let n and k be integers with $1 \leq k \leq n$, and let A be a set of n real numbers. For i with $1 \leq i \leq n$, let S_i be the set of all subsets of A with i elements, and let $\sigma_i = \sum_{s \in S_i} \max(s)$. Express the kth smallest element of A as a linear combination of $\sigma_1, \ldots, \sigma_n$.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Let $a_1 \leq a_2 \leq \cdots \leq a_n$ be the elements of A. Then $a_i = \max(s)$ iff $s = \{a_i\} \cup s'$ where $s' \subset \{a_1, \ldots, a_{i-1}\}$. Therefore

$$\sigma_k = \sum_{s \in S_k} \max(s) = \sum_{i=1}^{n} \binom{i-1}{k-1} a_i.$$

We claim that

$$a_k = \sum_{i=1}^{n} (-1)^{i+k} \binom{i-1}{k-1} \sigma_i.$$

This can be verified directly as follows:

$$\begin{aligned}
\sum_{k=1}^{n} \binom{i-1}{k-1} \cdot (-1)^{k+j} \binom{k-1}{j-1} &= \sum_{k=1}^{n} (-1)^{k+j} \frac{(i-1)!}{(k-1)!(i-k)!} \cdot \frac{(k-1)!}{(j-1)!(k-j)!} \cdot \frac{(i-j)!}{(i-j)!} \\
&= \binom{i-1}{j-1} \sum_{k=1}^{n} (-1)^{k-j} \binom{i-j}{k-j} [j \leq k \leq i] \\
&= [j \leq i] \binom{i-1}{j-1} \sum_{m=0}^{i-j} (-1)^m \binom{i-j}{m} = [i = j]
\end{aligned}$$

where $[Q]$ is 1 if Q is true and it is 0 otherwise. □