Problem 11519
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Find

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(-1)^{n+m}\frac{H_{n+m}}{n+m},$$

where H_n denotes the nth harmonic number.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

We first note that

$$\begin{aligned}\sum_{n=1}^{N}\sum_{m=1}^{\infty}(-1)^{n+m}\frac{H_{n+m}}{n+m} &= \sum_{n=1}^{N}\sum_{m=n+1}^{\infty}(-1)^{m}\frac{H_{m}}{m}\\ &= N\sum_{m=1}^{\infty}(-1)^{m}\frac{H_{m}}{m} - \sum_{n=1}^{N}\sum_{m=1}^{n}(-1)^{m}\frac{H_{m}}{m}\\ &= N\sum_{m=1}^{\infty}(-1)^{m}\frac{H_{m}}{m} - \sum_{m=1}^{N}\sum_{n=m}^{N}(-1)^{m}\frac{H_{m}}{m}\\ &= N\sum_{n=N+1}^{\infty}(-1)^{n}\frac{H_{n}}{n} + \sum_{n=1}^{N}(-1)^{n-1}\frac{H_{n}}{n} + \sum_{n=1}^{N}(-1)^{n}H_{n}.\end{aligned}$$

If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent series and $a_n = b_n + O(1/n^3)$ then

$$N\sum_{n=N+1}^{\infty} a_n = N\sum_{n=N+1}^{\infty} b_n + o(1).$$

Hence

$$(-1)^n\frac{H_n}{n} = (-1)^n\frac{\log(n)+\gamma+\frac{1}{2n}}{n} + O(1/n^3)$$

implies that

$$N\sum_{n=N+1}^{\infty}(-1)^n\frac{H_n}{n} = -(-1)^N\frac{\log(N)}{2} - (-1)^N\frac{\gamma}{2} + o(1)$$

because

$$\begin{aligned}\sum_{n=1}^{N}(-1)^n\frac{\log(n)}{n} &= \gamma\log(2) - \frac{\log^2(2)}{2} + (-1)^N\frac{\log(N)}{2N} + o(1/N),\\ \sum_{n=1}^{N}\frac{(-1)^n}{n} &= -\log(2) + \frac{(-1)^N}{2N} + o(1/N),\\ \sum_{n=1}^{N}\frac{(-1)^n}{n^2} &= -\frac{\pi^2}{12} + o(1/N).\end{aligned}$$

Moreover

$$\begin{aligned}\sum_{n=1}^{N}(-1)^{n-1}\frac{H_n}{n} &= \int_0^1 \frac{\log(1+x)}{x(1+x)}\,dx = \int_0^1\frac{\log(1+x)}{x}\,dx - \int_0^1\frac{\log(1+x)}{1+x}\,dx\\ &= \sum_{n=1}^{N}\frac{(-1)^{n-1}}{n^2} - \left[\frac{1}{2}\log^2(1+x)\right]_0^1 = \frac{\pi^2}{12} - \frac{\log^2(2)}{2},\end{aligned}$$

and

$$\sum_{n=1}^{N}(-1)^n H_n = \frac{1}{2}\sum_{n=1}^{N}\frac{(-1)^n}{n} + \frac{(-1)^N}{2}H_N = -\frac{\log(2)}{2} + (-1)^N\frac{\log(N)}{2} + (-1)^N\frac{\gamma}{2} + o(1).$$

Finally

$$\sum_{n=1}^{N}\sum_{m=1}^{\infty}(-1)^{n+m}\frac{H_{n+m}}{n+m} = \frac{\pi^2}{12} - \frac{\log^2(2)}{2} - \frac{\log(2)}{2} + o(1)$$

and the desired sum is

$$\frac{\pi^2}{12} - \frac{\log^2(2)}{2} - \frac{\log(2)}{2} \approx 0.2356669363.$$

□