Problem 11517
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Proposed by C. Lupu and T. Lupu (Romania).

Let f be a three-times differentiable real-valued function on $[a,b]$ with $f(a)=f(b)$. Prove that

$$\left|\int_a^{(a+b)/2} f(x)\,dx - \int_{(a+b)/2}^b f(x)\,dx\right| \leq \frac{(b-a)^4}{192}\sup_{x\in[a,b]}|f'''(x)|.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Let $c=(a+b)/2$ and let $P(x)$ be the quadratic polynomial which interpolates the points $(a,f(a))$, $(b,f(b))$, and $(c,f(c))$:

$$P(x)=f(a)\cdot\frac{(x-b)(x-c)}{(a-b)(a-c)}+f(b)\cdot\frac{(x-a)(x-c)}{(b-a)(b-c)}+f(c)\cdot\frac{(x-a)(x-b)}{(c-a)(c-b)}.$$

Then there is $\theta(x)\in[a,b]$ such that

$$f(x)=P(x)+\frac{f'''(\theta(x))}{6}(x-a)(x-b)(x-c).$$

Moreover

$$\int_a^c P(x)\,dx=\frac{(b-a)}{24}\left(5f(a)+8f(c)-f(b)\right)\,,\quad \int_c^b P(x)\,dx=\frac{(b-a)}{24}\left(-f(a)+8f(c)+5f(b)\right)$$

and since $f(a)=f(b)$ it follows that $\int_a^c P(x)\,dx-\int_c^b P(x)\,dx=0$. Hence

$$\begin{aligned}\left|\int_a^c f(x)\,dx-\int_c^b f(x)\,dx\right| &= \left|\int_a^c \frac{f'''(\theta(x))}{6}(x-a)(x-b)(x-c)\,dx-\int_c^b \frac{f'''(\theta(x))}{6}(x-a)(x-b)(x-c)\,dx\right|\\ &\leq \frac{1}{6}\sup_{x\in[a,b]}|f'''(x)|\int_a^b|(x-a)(x-b)(x-c)|\,dx=\frac{(b-a)^4}{192}\sup_{x\in[a,b]}|f'''(x)|.\end{aligned}$$

□