Problem 11515
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Find a closed-form expression for

$$\sum_{n=1}^{\infty} 4^n \sin^4(2^{-n}\theta).$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

We first show by induction that for non-negative integer N

$$\sum_{n=1}^{N} 4^n \sin^4(2^{-n}\theta) = 4^N \sin^2(2^{-N}\theta) - \sin^2(\theta).$$

The equality is trivial for $N = 0$. Assume that $N \geq 0$. Then

$$\begin{aligned}
\sum_{n=1}^{N+1} 4^n \sin^4(2^{-n}\theta) &= 4^N \sin^2(2^{-N}\theta) - \sin^2(\theta) + 4^{N+1} \sin^4(2^{-(N+1)}\theta) \\
&= 4^N \sin^2(2^{-N}\theta) - \sin^2(\theta) + 4^N (1 - \cos(2^{-N}\theta))^2 \\
&= 4^N (2 - 2\cos(2^{-N}\theta)) - \sin^2(\theta) \\
&= 4^{N+1} \sin^2(2^{-(N+1)}\theta)) - \sin^2(\theta).
\end{aligned}$$

Finally, by taking the limit as N goes to infinity we obtain

$$\sum_{n=1}^{\infty} 4^n \sin^4(2^{-n}\theta) = \lim_{N\to\infty} 4^N \sin^2(2^{-N}\theta) - \sin^2(\theta) = \theta^2 - \sin^2(\theta).$$

$\square$