Problem 11512
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Proposed by F. Holland (Ireland).

Let N be a nonnegative integer. For $x \geq 0$ prove that

$$\sum_{m=0}^{N} \frac{1}{m!} \left(\sum_{k=1}^{N-m+1} \frac{x^k}{k} \right)^m \geq 1 + x + \ldots + x^N.$$

Solution proposed by Paolo Perfetti and Roberto Tauraso,
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Let $P(x)$ denote the polynomial on the left-hand side of the inequality

$$\begin{aligned} P(x) &= 1 + \sum_{m=1}^{N} \frac{1}{m!} \sum_{k_1=1}^{N-m+1} \cdots \sum_{k_m=1}^{N-m+1} \frac{x^{k_1+\ldots+k_m}}{k_1 \cdot k_2 \cdots k_m} = \\ &= 1 + \sum_{m=1}^{N} \frac{1}{m!} \sum_{r=m}^{m(N-m+1)} x^r \sum_{\substack{1 \leq k_1, \ldots, k_m \\ k_1+\ldots+k_m = r}} \frac{1}{k_1 \cdot k_2 \cdots k_m}. \end{aligned}$$

It is easy to verify that the degree of $P(x)$ is $N(N+2)/4 \geq N$ if N is even and $(N+1)^2/4 \geq N$ if N is odd. Hence, since the coefficients of $P(x)$ are nonnegative, it suffices to show that the coefficients of the monomials of degree $0, 1, \ldots, N$ of $P(x)$, are equal to 1, that is

$$\frac{1}{n!} \frac{d^n}{dx^n} P(x) \Big|_{x=0} = 1 \qquad n = 0, 1, 2, \ldots, N.$$

For $n = 0$ and for any value of N it is trivial, so we assume that $1 \leq n \leq N$. Therefore

$$\frac{1}{n!} \frac{d^n}{dx^n} P(x) = \frac{1}{n!} \sum_{m=1}^{N} \frac{1}{m!} \sum_{r=m}^{m(N-m+1)} x^{r-n} \sum_{\substack{1 \leq k_1, \ldots, k_m \\ k_1+\ldots+k_m = r}} \frac{r(r-1)(r-2) \cdots (r-n+1)}{k_1 \cdot k_2 \cdots k_m}$$

and finally we have that

$$\begin{aligned} \frac{1}{n!} \frac{d^n}{dx^n} P(x) \Big|_{x=0} &= \sum_{m=1}^{N} \frac{1}{m!} \sum_{\substack{1 \leq k_1, \ldots, k_m \\ k_1+\ldots+k_m = n}} \frac{1}{k_1 \cdot k_2 \cdots k_m} \\ &= [x^n] \sum_{m=1}^{N} \frac{1}{m!} \log^m \left(\frac{1}{1-x} \right) = [x^n] \sum_{m=1}^{\infty} \frac{1}{m!} \log^m \left(\frac{1}{1-x} \right) \\ &= [x^n] \left(\exp \left(\log \left(\frac{1}{1-x} \right) \right) - 1 \right) = [x^n] \frac{x}{1-x} = 1. \end{aligned}$$

□