Problem 11509
(American Mathematical Monthly, Vol.117, June-July 2010)

Proposed by W. Stanford (USA).

Let m be a positive integer. Prove that

$$\sum_{k=m}^{m^2-m+1} \frac{\binom{m^2-2m+1}{k-m}}{k\binom{m^2}{k}} = \frac{1}{m\binom{2m-1}{m}}.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

We first consider identity (4.6) in *Combinatorial Identities* by H. W. Gould:

$$\sum_{k=0}^{n} \binom{n}{k}\binom{n+2t}{k+t}^{-1} = \frac{n+1+2t}{1+2t}\binom{2t}{t}.$$

Here is a short proof

$$\begin{aligned}
\sum_{k=0}^{n} \binom{n}{k}\binom{n+2t}{k+t}^{-1} &= \sum_{k=0}^{n} \frac{n!(n+t-k)!(k+t)!}{(n-k)!k!(n+2t)!} \\
&= \frac{n!(t!)^2}{(n+2t)!}\sum_{k=0}^{n}\binom{k+t}{k}\binom{n+t-k}{n-k} \\
&= \frac{n!(t!)^2}{(n+2t)!}\sum_{k=0}^{n}(-1)^k\binom{-(t+1)}{k}(-1)^{n-k}\binom{-(t+1)}{n-k} \\
&= \frac{n!(t!)^2}{(n+2t)!}(-1)^n\binom{-2(t+1)}{n} = \frac{n!(t!)^2}{(n+2t)!}\binom{n+2(t+1)-1}{n} = \frac{n+1+2t}{1+2t}\binom{2t}{t}.
\end{aligned}$$

The desired sum follows easily from the above identity by taking $n=(m-1)^2$ and $t=m-1$

$$\begin{aligned}
\sum_{k=m}^{m^2-m+1}\binom{m^2-2m+1}{k-m}\frac{1}{k}\binom{m^2}{k}^{-1} &= \sum_{k=m}^{m^2-m+1}\binom{m^2-2m+1}{k-m}\frac{1}{m^2}\binom{m^2-1}{k-1}^{-1} \\
&= \frac{1}{m^2}\sum_{k=0}^{(m-1)^2}\binom{(m-1)^2}{k}\binom{m^2-1}{k+m-1}^{-1} \\
&= \frac{1}{m^2}\cdot\frac{m^2}{2m-1}\binom{2(m-1)}{m-1}^{-1} = \frac{1}{m}\binom{2m-1}{m}.
\end{aligned}$$

□