Problem 11505
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Proposed by B. Burdick (USA).

Define $\{a_n\}$ to be the periodic sequence given by $a_1 = a_3 = 1$, $a_2 = 2$, $a_4 = a_6 = -1$, $a_5 = -2$, and $a_n = a_{n-6}$ for $n \geq 7$. Let $\{F_n\}$ be the Fibonacci sequence with $F_1 = F_2 = 1$. Show that

$$\sum_{k=1}^{\infty} \frac{a_k F_k F_{2k-1}}{2k-1} \sum_{n=0}^{\infty} \frac{(-1)^{kn}}{F_{kn+2k-1}F_{kn+3k-1}} = \frac{\pi}{4}.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

By d'Ocagne's identity

$$F_N F_{M+1} - F_{N+1} F_M = (-1)^M F_{N-M}$$

we have that for $M = kn + 2k - 1$ and $N = k(n+1) + 2k - 1$

$$\frac{(-1)^{kn}}{F_{kn+2k-1}F_{kn+3k-1}} = \frac{-F_{k(n+1)+2k-1}F_{kn+2k} + F_{k(n+1)+2k}F_{kn+2k-1}}{F_k F_{kn+2k-1} F_{k(n+1)+2k-1}}$$

$$= \frac{1}{F_k}\left(\frac{F_{k(n+1)+2k}}{F_{k(n+1)+2k-1}} - \frac{F_{kn+2k}}{F_{kn+2k-1}}.\right)$$

Hence

$$\sum_{n=0}^{\infty} \frac{(-1)^{kn}}{F_{kn+2k-1}F_{kn+3k-1}} = \frac{1}{F_k}\sum_{n=0}^{\infty}\left(\frac{F_{k(n+1)+2k}}{F_{k(n+1)+2k-1}} - \frac{F_{kn+2k}}{F_{kn+2k-1}}.\right) = \frac{1}{F_k}\left(\Phi - \frac{F_{2k}}{F_{2k-1}}\right)$$

where $\Phi = (\sqrt{5}+1)/2$. Therefore

$$S := \sum_{k=1}^{\infty} \frac{a_k F_k F_{2k-1}}{2k-1} \sum_{n=0}^{\infty} \frac{(-1)^{kn}}{F_{kn+2k-1}F_{kn+3k-1}} = \sum_{k=1}^{\infty} \frac{a_k(\Phi F_{2k-1} - F_{2k})}{2k-1} = \sum_{k=1}^{\infty} \frac{a_k(\Phi)^{-(2k-1)}}{2k-1}$$

because $F_{N+1} - \Phi F_N = (-\Phi)^{-N}$.
Since $a_k = 2\mathrm{Im}(\exp(\pi i/6)^{2k-1})$ then by letting $z = \exp(\pi i/6)/\Phi = (\sqrt{3}+i)/(2\Phi)$ we obtain

$$S = 2\mathrm{Im}\left(\sum_{k=1}^{\infty} \frac{1}{2k-1}\left(\frac{\exp(\pi i/6)}{\Phi}\right)^{2k-1}\right) = \mathrm{Im}\left(\ln\left(\frac{1+z}{1-z}\right)\right) = \mathrm{Arg}\left(\frac{1+z}{1-z}\right)$$

$$= \arctan\left(\frac{2\mathrm{Im}(z)}{1-|z|^2}\right) = \arctan\left(\frac{1/\Phi}{1-1/\Phi^2}\right) = \arctan(1) = \frac{\pi}{4}.$$

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