Problem 11494
(American Mathematical Monthly, Vol.117, March 2010)

Proposed by O. Furdui (Romania).

Let A denote the Glaisher-Kinkelin constant, given by

$$A = \lim_{n\to\infty} n^{-n^2/2-n/2-1/12} e^{n^2/4} \prod_{k=1}^{n} k^k = 1.2824\ldots$$

Prove that

$$\prod_{n=1}^{\infty} \left(\frac{n!}{\sqrt{2\pi n}(n/e)^n} \right)^{(-1)^{n-1}} = \frac{A^3}{2^{7/12}\pi^{1/4}}.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

We first note that the infinite product can be written as

$$\prod_{n=1}^{\infty} \left(\frac{(2n-1)!}{\sqrt{2\pi(2n-1)}((2n-1)/e)^{2n-1}} \right) \cdot \left(\frac{(2n)!}{\sqrt{2\pi 2n}(2n/e)^{2n}} \right)^{-1} = \prod_{n=1}^{\infty} \frac{1}{e} \left(\frac{2n}{2n-1} \right)^{2n-1/2}.$$

Hence it suffices to prove that

$$S_N := \sum_{n=1}^{N} \left(-1 + (2n - \frac{1}{2})(\log(2n) - \log(2n-1)) \right) = 3\log(A) - \frac{7}{12}\log(2) - \frac{1}{4}\log(\pi) + o(1).$$

By expanding the partial sum we obtain

$$\begin{aligned} S_N &= -N + \sum_{n=1}^{N}(2n)\log(2n) - \frac{1}{2}\sum_{n=1}^{N}\log(2n) - \sum_{n=1}^{N}(2n-1)\log(2n-1) - \frac{1}{2}\sum_{n=1}^{N}\log(2n-1) \\ &= -N + 2\sum_{n=1}^{N}(2n)\log(2n) - \sum_{n=1}^{2N} n\log(n) + \frac{1}{2}\sum_{n=1}^{2N}\log(n) \\ &= -N + 4\log(2)\sum_{n=1}^{N} n + 4\sum_{n=1}^{N} n\log(n) - \sum_{n=1}^{2N} n\log(n) - \frac{1}{2}\sum_{n=1}^{2N}\log(n). \end{aligned}$$

By Stirling's approximation

$$\sum_{n=1}^{2N}\log(n) = \frac{1}{2}\log(2\pi) + \frac{1}{2}\log(2N) + (2N)\log(2N) - 2N + o(1)$$

and by the constant A's definition we have that

$$\sum_{n=1}^{N} n\log(n) = \log(A) + \left(\frac{N^2}{2} + \frac{N}{2} + \frac{1}{12} \right)\log(N) - \frac{N^2}{4} + o(1)$$

and

$$\sum_{n=1}^{2N} n\log(n) = \log(A) + \left(2N^2 + N + \frac{1}{12} \right)\log(2N) - N^2 + o(1).$$

Finally,

$$\begin{aligned}S_N &= -N + 2\log(2)N(N+1) + 4\log(A) + \left(2N^2 + 2N + \frac{1}{3}\right)\log(N) - N^2\\&- \log(A) - \left(2N^2 + N + \frac{1}{12}\right)\log(2N) + N^2 - \frac{1}{4}\log(2\pi)\\&- \frac{1}{4}\log(2N) - N\log(2N) + N + o(1) = 3\log(A) - \frac{7}{12}\log(2) - \frac{1}{4}\log(\pi) + o(1).\end{aligned}$$

□