Problem 11485
(American Mathematical Monthly, Vol.117, February 2010)

Proposed by N. Badhoniya, K. S. Bhanu, and M. N. Deshpande (India).

An urn contains a white balls and b black balls, and $a \geq 2b+3$. Balls are drawn at random from the urn and placed in a row as they are drawn. Drawings halts when three white balls are drawn in succession. Let X be the number of paired white balls in the lineup produced during play, and let Y be the number of isolated white balls. Show that

$$E[X] = \frac{b}{a+1}, \quad E[Y] = \frac{b(a+b+1)}{(a+1)(a+2)}.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

The generating function which counts the number of paired white balls w_2 and the number of isolated white balls w_1 before the first group of three white balls is

$$f(x,y) = \sum_{b_1\geq 0}\sum_{w_1\geq 0}\sum_{w_3\geq 0} \binom{b_1}{w_1+w_2}\binom{w_1+w_2}{w_2}\binom{a+b-(b_1+w_1+2w_2+3)}{b-b_1} x^{w_2}y^{w_1}.$$

where we adopt the convention that the binomial coefficient $\binom{n}{k}=0$ if the condition $n \geq k \geq 0$ is not satisfied. Then

$$f_x(a,b) := \left.\frac{\partial f}{\partial x}\right|_{x=y=1} = E[X]\binom{a+b}{b}, \quad \text{and} \quad f_y(a,b) := \left.\frac{\partial f}{\partial y}\right|_{x=y=1} = E[Y]\binom{a+b}{b}.$$

Since

$$\binom{a+(b+1)-(b_1+w_1+2w_2+3)}{(b+1)-b_1} = \binom{a+b-(b_1+w_1+2w_2+3)}{b-b_1} + \binom{(a-1)+(b+1)-(b_1+w_1+2w_2+3)}{(b+1)-b_1}$$

then we have the recurrences

$$f_x(a,b+1) = f_x(a,b)+f_x(a-1,b+1) \quad \text{and} \quad f_y(a,b+1) = f_y(a,b)+f_y(a-1,b+1).$$

Moreover, it is easy to verify that

$$f_x(0,b)=b, \quad f_x(a,0)=0, \quad f_y(0,b)=\frac{b(b+1)}{2}, \quad f_y(a,0)=0.$$

The recurrences and the initial conditions uniquely determine $f_x(a,b)$ and $f_y(a,b)$:

$$f_x(a,b) = \frac{b}{a+1}\binom{a+b}{b} \quad \text{and} \quad f_y(a,b) = f_x(a+1,b) = \frac{b}{a+2}\binom{a+b+1}{a+1} = \frac{b(a+b+1)}{(a+1)(a+2)}\binom{a+b}{b}$$

which yield the desired formulas for $E[X]$ and $E[Y]$. □