Problem 11477
(American Mathematical Monthly, Vol.117, January 2010)

Proposed by A. Gonzalez (Spain) and J. H. Nieto (Venezuela).

Several boxes sit in a row, numbered from 0 *on the left to* n *on the right. A frog hops from box to box, starting at time* 0 *in box* 0. *If at time* t*, the frog is in box* k*, it hops one box to the left with probability* k/n *and one box to the right with probability* $1-k/n$. *Let* $p_i(k)$ *be the probability that the frog launches its* $(t+1)$*th hop from box* k. *Find* $\lim_{i\to\infty} p_{2i}(k)$ *and* $\lim_{i\to\infty} p_{2i+1}(k)$.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

This process can be described as a finite Markov chain with states $\{0,1,2,\ldots,n\}$ (the boxes), with a transition matrix P such that

$$p_{k,k+1}=1-k/n, \quad p_{k,k-1}=k/n \quad \text{and} \quad p_{k,j}=0 \text{ otherwise.}$$

Starting from an even-numbered state, the process can be in even-numbered states only in an even number of steps, and in an odd-numbered state in an odd number of steps. Hence the even and odd states form two cyclic classes. Let Q_e be the restriction of P^2 to the even-numbered state. It is easy to see that this transition matrix is *regular* (i. e. some power of the matrix has all nonzero entries) and therefore there exists a unique invariant probability vector u_e (i. e. $u_eQ_e=u_e$) such that

$$\lim_{n\to\infty} v_e Q_e^n = u_e$$

for any probability vectors v_e. Similarly, letting Q_o be the restriction of P^2 to the odd-numbered state then there exists a unique invariant probability vector u_o such that

$$\lim_{n\to\infty} v_o Q_o^n = u_o$$

for any probability vectors v_o. This implies that the limits $q_e(n,k):=\lim_{i\to\infty} p_{2i}(k)$ and $q_o(n,k):=\lim_{i\to\infty} p_{2i+1}(k)$ exist and, since the initial state is 0, they are the components of two unique probability vectors such that: $q_e(n,k)=0$ for k odd, $q_o(n,k)=0$ for k even, and they satisfy the equations

$$\begin{aligned} q_o(n,k) &= \left(1-\frac{k-1}{n}\right) q_e(n,k-1)+\frac{k+1}{n} q_e(n,k+1),\\ q_e(n,k) &= \left(1-\frac{k-1}{n}\right) q_o(n,k-1)+\frac{k+1}{n} q_o(n,k+1). \end{aligned}$$

(notice that $q_e(n,k)=q_e(n,k)=0$ for $k\not\in\{0,1,\ldots,n\}$). Finally, we can easily verify that

$$q_e(n,k)=\frac{1}{2^{n-1}}\binom{n}{k}[k \text{ is even}] \quad \text{and} \quad q_o(n,k)=\frac{1}{2^{n-1}}\binom{n}{k}[k \text{ is odd}].$$

In fact $\sum_{k=0}^n \binom{n}{k}[k \text{ is even}]=\sum_{k=0}^n \binom{n}{k}[k \text{ is odd}]=2^{n-1}$ and

$$\left(1-\frac{k-1}{n}\right)\binom{n}{k-1}+\frac{k+1}{n}\binom{n}{k+1}=\binom{n}{k-1}-\binom{n-1}{k-2}+\binom{n-1}{k}=\binom{n-1}{k-1}+\binom{n-1}{k}=\binom{n}{k}.$$

□