Problem 11472
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Proposed by M. Makhul (Iran).

Let t be a nonnegative integer, and let f be a $4t+3$ times continuously differentiable function on $\mathbb{R}$. Show that there is a number a such that

$$\prod_{k=0}^{4t+3} f^{(k)}(a) \geq 0.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

If at least one of the functions $f^{(k)}(x)$ vanishes at some point a, then we are done. Otherwise by the Intermediate Value Theorem, each $f^{(k)}(x)$ is either strictly positive or strictly negative on $\mathbb{R}$. By replacing $f(x)$ by $-f(x)$ if necessary, we may assume the derivative $f^{(4t+2)}(x) > 0$.
Since $4t+3$ is odd, by replacing $f(x)$ by $f(-x)$ if necessary, we may assume that also $f^{(4t+3)}(x) > 0$. Since the number of factors, i. e. $4t+4$, is a multiple of 4, these substitutions do not change the sign of the whole product.

Now we prove the following claim: if $g \in C^2(0,+\infty)$, and g', g'' are positive in $(a,+\infty)$ for $a \geq 0$ then there is a real number $b \geq a$ such that $g(x) > 0$ for all $x \in (b,+\infty)$.
Since g', g'' are positive then g' is positive and increasing. Thus for $x > a$

$$g(x) = g(a) + \int_a^x g'(s)\, ds \geq g(a) + (x-a)g'(a).$$

and it follows that $g(x) > 0$ for $x > b := \max(a, a - g(a)/g'(a))$.

Hence, by letting $a_{4t+3} = a_{4t+2} = 0$ and by using our claim for $g = f^{(4t+1)}$ we find that $f^{(4t+1)}(x) > 0$ for all $x \in (a_{4t+1}, +\infty)$ for some $a_{4t+1} \geq a_{4t+2}$. Now we can apply our claim for $g = f^{(4t)}$ and therefore, by iterating this process we have that for any integer $0 \leq k \leq 4t+3$

$$f^{(k)}(x) > 0 \text{ for all } x \in (a_0, +\infty) \text{ for some } a_0 \geq a_1 \geq \cdots \geq a_{4t+3} = a_{4t+2} = 0$$

and ffinally for $a \in (a_0, +\infty)$

$$\prod_{k=0}^{4t+3} f^{(k)}(a) > 0.$$

□

Remark. This problem is a generalization of the problem A3 of the The Fifty-Ninth Putnam Competition (1998).