Problem 11464
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Proposed by D. Beckwith (USA).

Let $a(n)$ be the number of ways to place n identical balls into a sequence of urns $U_1, U_2, \dots$ in such a way that U_1 receives at least one ball, and while any balls remain, each successive urn receives at least as many balls as in all the previous urns combined. Let $b(n)$ denote the number of partitions of n into powers of 2, with repeated powers allowed. Prove that $a(n) = b(n)$ for all $n \in \mathbb{N}$.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

The generating function of the sequence $\{b_n\}$ is

$$B(z) = \prod_{k=0}^{\infty} \frac{1}{1 - z^{2^k}}.$$

Let $a_k(n)$ be the number of ways we can put n balls into k urns.
Let u_i be the number of balls which goes in urn U_i for $i = 1, \dots, k$, then

$$\begin{aligned}
u_1 &= 1 + x_1 \\
u_2 &= u_1 + x_2 = 1 + x_1 + x_2 \\
u_3 &= u_1 + u_2 + x_3 = 2 + 2x_1 + x_2 + x_3 \\
&\cdots \\
u_k &= u_{k-1} + x_k = 2^{k-2} + 2^{k-2}x_1 + 2^{k-3}x_2 + \cdots + 2x_{k-2} + x_{k-1} + x_k
\end{aligned}$$

with $x_i \geq 0$. Hence for $n, k \geq 1$

$$n = u_1 + u_2 + u_3 + \cdots + u_k = 2^{k-1} + 2^{k-1}x_1 + 2^{k-2}x_2 + \cdots + 4x_{k-2} + 2x_{k-1} + x_k,$$

and

$$\sum_{n\geq 1} a_k(n) z^n = \sum_{x_1,\dots,x_k \geq 0} z^{u_1+\cdots+u_k} = \frac{z^{2^{k-1}}}{(1-z)(1-z^2)(1-z^4)\cdots(1-z^{2^{k-1}})}.$$

Since $a(n) = \sum_{k\geq 1} a_k(n)$, the generating function of the sequence $\{a_n\}$ is

$$A(z) = 1 + \sum_{n\geq 1} a(n) z^n = 1 + \sum_{k\geq 1} \frac{z^{2^{k-1}}}{(1-z)(1-z^2)(1-z^4)\cdots(1-z^{2^{k-1}})}.$$

Moreover, it can be verified by induction that for any positive integer N

$$1 + \sum_{k=1}^{N} \frac{z^{2^{k-1}}}{(1-z)(1-z^2)(1-z^4)\cdots(1-z^{2^{k-1}})} = \prod_{k=0}^{N-1} \frac{1}{1 - z^{2^k}}.$$

Hence $A(z) = B(z)$ and it follows that $a(n) = b(n)$ for all $n \in \mathbb{N}$. □