Problem 11464

Proposed by D. Beckwith (USA).

Let \( a(n) \) be the number of ways to place \( n \) identical balls into a sequence of urns \( U_1, U_2, \ldots \) in such a way that \( U_1 \) receives at least one ball, and while any balls remain, each successive urn receives at least as many balls as in all the previous urns combined. Let \( b(n) \) denote the number of partitions of \( n \) into powers of 2, with repeated powers allowed. Prove that \( a(n) = b(n) \) for all \( n \in \mathbb{N} \).

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma “Tor Vergata”, via della Ricerca Scientifica, 00133 Roma, Italy.

The generating function of the sequence \( \{ b_n \} \) is

\[
B(z) = \prod_{k=0}^\infty \frac{1}{1 - z^{2^k}}.
\]

Let \( a_k(n) \) be the number of ways we can put \( n \) balls into \( k \) urns. Let \( u_i \) be the number of balls which goes in urn \( U_i \) for \( i = 1, \ldots, k \), then

\[
\begin{align*}
  u_1 &= 1 + x_1 \\
  u_2 &= u_1 + x_2 = 1 + x_1 + x_2 \\
  u_3 &= u_1 + u_2 + x_3 = 2 + 2x_1 + x_2 + x_3 \\
  & \quad \ldots \\
  u_k &= u_{k-1} + x_k = 2^{k-2} + 2^{k-2}x_1 + 2^{k-3}x_2 + \cdots + 2x_{k-2} + x_{k-1} + x_k
\end{align*}
\]

with \( x_i \geq 0 \). Hence for \( n, k \geq 1 \)

\[
n = u_1 + u_2 + \ldots + u_k = 2^{k-1} + 2^{k-1}x_1 + 2^{k-2}x_2 + \cdots + 4x_{k-2} + 2x_{k-1} + x_k,
\]

and

\[
\sum_{n \geq 1} a_k(n)z^n = \sum_{x_1, \ldots, x_k \geq 0} z^{u_1+\cdots+u_k} = \frac{z^{2^{k-1}}}{(1-z)(1-z^2)(1-z^4)\cdots(1-z^{2^{k-1}})}.
\]

Since \( a(n) = \sum_{k \geq 1} a_k(n) \), the generating function of the sequence \( \{ a_n \} \) is

\[
A(z) = 1 + \sum_{n \geq 1} a(n)z^n = 1 + \sum_{k \geq 1} \frac{z^{2^{k-1}}}{(1-z)(1-z^2)(1-z^4)\cdots(1-z^{2^{k-1}})}.
\]

Moreover, it can be verified by induction that for any positive integer \( N \)

\[
1 + \sum_{k=1}^N \frac{z^{2^{k-1}}}{(1-z)(1-z^2)(1-z^4)\cdots(1-z^{2^{k-1}})} = \prod_{k=0}^{N-1} \frac{1}{1 - z^{2^k}}.
\]

Hence \( A(z) = B(z) \) and it follows that \( a(n) = b(n) \) for all \( n \in \mathbb{N} \). \( \square \)