Problem 11456
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Proposed by R. Mortini (France).

Find

$$\lim_{n\to\infty} n \prod_{m=1}^{n} \left(1 - \frac{1}{m} + \frac{5}{4m^2}\right).$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Let $z = -\frac{1}{2} + i$ then

$$e^{\gamma z} \prod_{m=1}^{n} \left(1 + \frac{z}{m}\right) e^{-z/m} \cdot e^{\gamma \overline{z}} \prod_{m=1}^{n} \left(1 + \frac{\overline{z}}{m}\right) e^{-\overline{z}/m} = e^{H_n - \gamma} \prod_{m=1}^{n} \left|1 + \frac{z}{m}\right|^2 = \frac{e^{H_n-\gamma}}{n}\, n \prod_{m=1}^{n} \left(1 - \frac{1}{m} + \frac{5}{4m^2}\right)$$

where $H_n = \sum_{k=1}^{n} \frac{1}{k}$. Since $H_n = \log n + \gamma + o(1)$ then $e^{H_n-\gamma}/n \to 1$.
Moreover, by Gamma function's definition

$$\frac{1}{z\Gamma(z)} = e^{\gamma z} \prod_{m=1}^{\infty} \left(1 + \frac{z}{m}\right) e^{-z/m}.$$

Hence

$$\lim_{n\to\infty} n \prod_{m=1}^{n} \left(1 - \frac{1}{m} + \frac{5}{4m^2}\right) = \frac{1}{z\Gamma(z)} \cdot \frac{1}{\overline{z}\Gamma(\overline{z})} = \frac{1}{\Gamma(z+1)} \cdot \frac{1}{\Gamma(\overline{z}+1)}.$$

Finally, since $z + 1 = \frac{1}{2} + i$ and $\overline{z} + 1 = \frac{1}{2} - i = 1 - (z + 1)$, by the reflection property

$$\lim_{n\to\infty} n \prod_{m=1}^{n} \left(1 - \frac{1}{m} + \frac{5}{4m^2}\right) = \frac{\sin(\pi(z+1))}{\pi} = \frac{\cos(i\pi)}{\pi} = \frac{\cosh(\pi)}{\pi}.$$

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