**Problem 11449**
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Proposed by M. Bataille (France).

*Find the maximum and minimum values of*

$$\frac{(a^3+b^3+c^3)^2}{(b^2+c^2)(c^2+a^2)(a^2+b^2)}$$

*given that* $a+b \geq c > 0$, $b+c \geq a > 0$, *and* $c+a \geq b > 0$.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

We will show that the maximum value is 2 and the minimum value is 9/8.
We will use the following notation:

$$[\alpha,\beta,\gamma] = \sum_{\text{sym}} x^{\alpha}\, y^{\beta}\, z^{\gamma}.$$

The maximum is attained for $a = 2$ and $b = c = 1$, moreover by letting $a = y + z$, $b = x + z$ and $c = x + y$ for $x, y, x \geq 0$ we have that the inequality

$$\frac{(a^3+b^3+c^3)^2}{(b^2+c^2)(c^2+a^2)(a^2+b^2)} \leq 2$$

is equivalent to

$$3[4,2,0] + 150[3,2,1] + 45[4,1,1] + 29[2,2,2] \geq 3[3,3,0]$$

which holds because $[4,2,0] \geq [3,3,0]$ by Muirhead's inequality.
The minimum is attained for $a = b = c = 1$, moreover by letting $a = y + z$, $b = x + z$ and $c = x + y$ for $x, y, x \geq 0$ we have that the inequality

$$\frac{(a^3+b^3+c^3)^2}{(b^2+c^2)(c^2+a^2)(a^2+b^2)} \geq \frac{9}{8}$$

is equivalent to

$$7[6,0,0] + 42[5,1,0] + 69[4,2,0] + 50[3,3,0] \geq 36[4,1,1] + 120[3,2,1] + 12[2,2,2]$$

which holds because by Muirhead's inequality:

$$\begin{aligned}
&7[6,0,0] + 29[5,1,0] \geq 36[4,1,1],\\
&13[5,1,0] + 69[4,2,0] + 38[3,3,0] \geq 120[3,2,1],\\
&12[3,3,0] \geq 12[2,2,2].
\end{aligned}$$

□