Problem 11448
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Proposed by Wei-Dong Jiang, China.

Let a, b, c be the side-lengths of a triangle, and let α, β, γ respectively denote half the measures of the angles opposite those sides. Show that

$$\frac{a}{b+c}\tan^2\beta\tan^2\gamma+\frac{b}{c+a}\tan^2\gamma\tan^2\alpha+\frac{c}{a+b}\tan^2\alpha\tan^2\beta\leq\frac{1}{6}.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Since

$$\tan^2\alpha=\frac{(s-b)(s-c)}{s(s-a)},\quad \tan^2\beta=\frac{(s-a)(s-c)}{s(s-b)},\quad \tan^2\gamma=\frac{(s-a)(s-b)}{s(s-c)}$$

where $s=(a+b+c)/2$, then the inequality becomes

$$\frac{a(s-a)^2}{(b+c)}+\frac{b(s-b)^2}{(c+a)}+\frac{c(s-c)^2}{(a+b)}\leq\frac{s^2}{6}$$

and letting $a=(y+z)/2$, $b=(z+x)/2$ and $c=(x+y)/2$ we obtain

$$\frac{(y+z)x^2}{2x+y+z}+\frac{(z+x)y^2}{x+2y+z}+\frac{(x+y)z^2}{x+y+2z}\leq\frac{(x+y+z)^2}{6}.$$

Note that the new variables x, y, z are non negative.
Clearing the denominators and simplifying we get

$$\sum_{\text{sym}}x^5y^0z^0+5\sum_{\text{sym}}x^4y^1z^0+6\sum_{\text{sym}}x^3y^1z^1\geq 7\sum_{\text{sym}}x^3y^2z^0+5\sum_{\text{sym}}x^2y^2z^1.$$

By Shur's inequality, for any real r we have that

$$\sum_{\text{cyc}}x^r(x-y)(x-z)\geq 0$$

Therefore for $r=3$ we obtain

$$\sum_{\text{sym}}x^5y^0z^0\geq 2\sum_{\text{sym}}x^4y^1z^0-\sum_{\text{sym}}x^3y^1z^1.$$

Hence

$$\sum_{\text{sym}}x^5y^0z^0+5\sum_{\text{sym}}x^4y^1z^0+6\sum_{\text{sym}}x^3y^1z^1\geq 7\sum_{\text{sym}}x^4y^1z^0+5\sum_{\text{sym}}x^3y^1z^1\geq 7\sum_{\text{sym}}x^3y^2z^0+5\sum_{\text{sym}}x^2y^2z^1.$$

which holds because by Muirhead's inequality

$$\sum_{\text{sym}}x^4y^1z^0\geq\sum_{\text{sym}}x^3y^2z^0\quad\text{and}\quad\sum_{\text{sym}}x^3y^1z^1\geq\sum_{\text{sym}}x^2y^2z^1.$$

Moreover it follows that equality holds iff $x=y=z$ that is when the triangle is equilateral. $\square$