Problem 11435
(American Mathematical Monthly, Vol.116, May 2009)

Proposed by P. Ligouras (Italy).

In a triangle T, let a, b, and c be the lenghts of the sides, r the inradius, and R be the circumradius. Show that

$$\frac{a^2bc}{(a+b)(a+c)} + \frac{b^2ca}{(b+c)(b+a)} + \frac{c^2ab}{(c+a)(c+b)} \leq \frac{9}{2}rR.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Note: I replaced 9/4 of the published statement with 9/2, otherwise the inequality does not hold.

Since

$$r = \frac{2A}{a+b+c},\ \ R = \frac{abc}{4A}$$

and letting $a = (y+z)/2$, $b = (x+z)/2$ and $c = (x+y)/2$ we obtain

$$\frac{y+z}{(x+2y+z)(x+y+2z)} + \frac{x+z}{(2x+y+z)(x+y+2z)} + \frac{x+y}{(2x+y+z)(x+2y+z)} \leq \frac{9}{8(x+y+z)}$$

Clearing the denominators and simplyfing we get

$$\sum_{\text{sym}} x^2y^1z^0 \leq \sum_{\text{sym}} x^3y^0z^0$$

which holds by Muirhead's inequality. Equality holds iff $x = y = z$ that is when the triangle is equilateral. $\square$