Problem 11428
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Proposed by W. Blumberg (USA).

Let p a prime congruent to 3 mod 4, and let a and q be integers, such that p does not divide q. Show that

$$\sum_{k=1}^{p} \lfloor (qk^2+a)/p \rfloor = 2a+1+\sum_{k=1}^{p} \lfloor (qk^2-a-1)/p \rfloor .$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Let $r_p(n)$ be the remainder of the division of n by p then, since $n = p\lfloor n/p \rfloor + \mathrm{r}_p(n)$, the above equality is equivalent to

$$\sum_{k=1}^{p} r_p(qk^2+a) = \sum_{k=1}^{p} r_p(qk^2-a-1).$$

Moreover

$$r_p(qk^2+a) = r_p(qk^2)+r_p(a) - p\,x(k) \quad \text{and} \quad r_p(qk^2-a-1) = r_p(qk^2)-r_p(a)-1+p\,y(k)$$

with

$$x(k) = [r_p(qk^2) \geq p - r_p(a)] \quad \text{and} \quad y(k) = [r_p(qk^2) \leq r_p(a)]$$

where $[Q]$ is equal to 1 if proposition Q is true and it is 0 otherwise.
Hence, since $x(p) = 0$ and $y(p) = 1$ then it suffices to show that

$$\sum_{k=1}^{p-1} (x(k)+y(k)) = 2r_p(a).$$

Note that $r_p(qk^2) = j$ for some $j \in \{1, \ldots, p-1\}$ iff $k^2 = jq^{-1}$ mod p (q is invertible mod p), that is iff jq^{-1} is a square mod p which means that $\left(\frac{jq^{-1}}{p}\right) = 1$ where $\left(\frac{\cdot}{p}\right)$ is the Legendre symbol. Therefore the number of $k \in \{1, \ldots, p-1\}$ such that $r_p(qk^2) = j$ is $\left(\frac{jq^{-1}}{p}\right) + 1$ (it gives 0 or 2). Since $p = 3$ mod 4 then $\left(\frac{-1}{p}\right) = (-1)^{(p-1)/2} = -1$, and it follows that

$$\sum_{k=1}^{p-1} x(k) = \sum_{j=p-r_p(a)}^{p-1} \left(\left(\frac{jq^{-1}}{p}\right) + 1 \right) = r_p(a) + \left(\frac{q^{-1}}{p}\right) \sum_{j=1}^{r_p(a)} \left(\frac{p-j}{p}\right) = r_p(a) - \left(\frac{q^{-1}}{p}\right) \sum_{j=1}^{r_p(a)} \left(\frac{j}{p}\right).$$

In a similar way

$$\sum_{k=1}^{p-1} y(k) = \sum_{j=1}^{r_p(a)} \left(\left(\frac{jq^{-1}}{p}\right) + 1 \right) = r_p(a) + \left(\frac{q^{-1}}{p}\right) \sum_{j=1}^{r_p(a)} \left(\frac{j}{p}\right).$$

Finally,

$$\sum_{k=1}^{p-1} (x(k)+y(k)) = r_p(a) - \left(\frac{q^{-1}}{p}\right) \sum_{j=1}^{r_p(a)} \left(\frac{j}{p}\right) + r_p(a) + \left(\frac{q^{-1}}{p}\right) \sum_{j=1}^{r_p(a)} \left(\frac{j}{p}\right) = 2r_p(a).$$

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