Problem 11426
(American Mathematical Monthly, Vol.116, April 2009)

Proposed by M. L. Glasser (USA).

Find

$$\frac{\Gamma(1/14)\Gamma(9/14)\Gamma(11/14)}{\Gamma(3/14)\Gamma(5/14)\Gamma(13/14)}.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

We will prove the following generalization: let p be a prime such that $p = 7 \bmod 8$ then

$$Q(p) := \prod_{a\in O(2p)} \left(\Gamma\left(\frac{a}{2p}\right)\right)^{\left(\frac{a}{p}\right)} = 2^{\sum_{k=1}^{(p-1)/2}\left(\frac{k}{p}\right)}$$

where $O(s)$ is the set of odd positive integers less than s and $\left(\frac{\cdot}{p}\right)$ is the Legendre symbol. In the particular case when $p = 7$ we have that

$$Q(7) = \frac{\Gamma(1/14)\Gamma(9/14)\Gamma(11/14)}{\Gamma(3/14)\Gamma(5/14)\Gamma(13/14)} = 2^{\left(\frac{1}{7}\right)+\left(\frac{2}{7}\right)+\left(\frac{3}{7}\right)} = 2.$$

Since $p = 7 \bmod 8$

$$\left(\frac{2p-a}{p}\right) = \left(\frac{-a}{p}\right) = \left(\frac{-1}{p}\right)\left(\frac{a}{p}\right) = (-1)^{(p-1)/2}\left(\frac{a}{p}\right) = -\left(\frac{a}{p}\right)$$

then

$$Q(p) = \prod_{a\in O(p)} \left(\frac{\Gamma(a/2p)}{\Gamma((2p-a)/2p)}\right)^{\left(\frac{a}{p}\right)}.$$

$\Gamma(x)$ satisfies the *reflection formula* and the *duplication formula*:

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}\ , \qquad \Gamma(x)\Gamma\left(x+\frac{1}{2}\right) = \Gamma(2x)2^{1-2x}\sqrt{\pi},$$

therefore we have that for $x \neq 1, 1/2$

$$\begin{aligned}
\frac{\Gamma(x)}{\Gamma(1-x)} &= \Gamma(x)^2\,\frac{\sin(\pi x)}{\pi} = \frac{\Gamma(2x)^2}{\Gamma\left(x+\frac{1}{2}\right)^2}\,\frac{\sin(\pi(1-2x))}{\sin(\pi/2(1-2x))}\,2^{1-4x} \\
&= \frac{\Gamma(2x)^2}{\Gamma\left(x+\frac{1}{2}\right)^2}\,\frac{\Gamma\left(x+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}-x\right)}{\Gamma(2x)\Gamma(1-2x)}\,2^{1-4x} \\
&= \frac{\Gamma(2x)\Gamma\left(\frac{1}{2}-x\right)}{\Gamma\left(x+\frac{1}{2}\right)\Gamma(1-2x)}\,2^{1-4x}.
\end{aligned}$$

Hence

$$Q(p) = \prod_{a\in O(p)} 2^{\left(1-\frac{2a}{p}\right)\left(\frac{a}{p}\right)} \left(\frac{\Gamma\left(\frac{a}{p}\right)\Gamma\left(\frac{\frac{1}{2}(p-a)}{p}\right)}{\Gamma\left(\frac{p-a}{p}\right)\Gamma\left(\frac{\frac{1}{2}(p+a)}{p}\right)}\right)^{\left(\frac{a}{p}\right)}.$$

Moreover, since $p = 7 \bmod 8$ and a is an odd integer non divisible by p

$$\left(\frac{\frac{1}{2}(p\pm a)}{p}\right) = \left(\frac{2}{p}\right)^{-1}\left(\frac{p\pm a}{p}\right) = (-1)^{-(p^2-1)/8}\left(\frac{\pm a}{p}\right) = \pm\left(\frac{a}{p}\right).$$

So

$$
\begin{aligned}
Q(p) &= \prod_{a\in O(p)} 2^{\left(1-\frac{2a}{p}\right)\left(\frac{a}{p}\right)} \frac{\left(\Gamma\left(\frac{a}{p}\right)\right)^{\left(\frac{a}{p}\right)}\left(\Gamma\left(\frac{p-a}{p}\right)\right)^{\left(\frac{p-a}{p}\right)}}{\left(\Gamma\left(\frac{\frac{1}{2}(p+a)}{p}\right)\right)^{\left(\frac{\frac{1}{2}(p+a)}{p}\right)}\left(\Gamma\left(\frac{\frac{1}{2}(p-a)}{p}\right)\right)^{\left(\frac{\frac{1}{2}(p-a)}{p}\right)}} \\
&= \prod_{a\in O(p)} 2^{\left(1-\frac{2a}{p}\right)\left(\frac{a}{p}\right)} \frac{\prod_{k=1}^{p-1}\left(\Gamma\left(\frac{k}{p}\right)\right)^{\left(\frac{k}{p}\right)}}{\prod_{k=1}^{p-1}\left(\Gamma\left(\frac{k}{p}\right)\right)^{\left(\frac{k}{p}\right)}} = 2^{\sum_{a\in O(p)}\left(1-\frac{2a}{p}\right)\left(\frac{a}{p}\right)}.
\end{aligned}
$$

Since $\left(\frac{2}{p}\right) = 1$ and $\left(\frac{-1}{p}\right) = -1$ then we can simplify the exponent:

$$
\sum_{a\in O(p)} \left(\frac{a}{p}\right) = \sum_{k=1}^{(p-1)/2} \left(\frac{p-2k}{p}\right) = -\sum_{k=1}^{(p-1)/2} \left(\frac{k}{p}\right),
$$

and

$$
\begin{aligned}
\sum_{a\in O(p)} \frac{a}{p}\left(\frac{a}{p}\right) &= \sum_{k=1}^{p-1} \frac{k}{p}\left(\frac{k}{p}\right) - \sum_{k=1}^{(p-1)/2} \frac{2k}{p}\left(\frac{2k}{p}\right) \\
&= \sum_{k=(p+1)/2}^{p-1} \frac{k}{p}\left(\frac{k}{p}\right) - \sum_{k=1}^{(p-1)/2} \frac{k}{p}\left(\frac{k}{p}\right) \\
&= \sum_{k=1}^{(p-1)/2} \frac{p-k}{p}\left(\frac{p-k}{p}\right) - \sum_{k=1}^{(p-1)/2} \frac{k}{p}\left(\frac{k}{p}\right) = -\sum_{k=1}^{(p-1)/2} \left(\frac{k}{p}\right).
\end{aligned}
$$

Finally

$$
Q(p) = 2^{\sum_{a\in O(p)}\left(1-\frac{2a}{p}\right)\left(\frac{a}{p}\right)} = 2^{\sum_{k=1}^{(p-1)/2}\left(\frac{k}{p}\right)}.
$$

$\square$