

Problem 11424

(American Mathematical Monthly, Vol.116, March 2009)

Proposed by E. Deutsch (USA).

Find the number x_n of bit strings of length n in which the number of 00 substrings is equal to the number of 11 substrings.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

A bit strings of length n is an ordered collection of $1 \leq m \leq n$ blocks each of length $b_r + 1 \geq 1$ of consecutive 0's and 1's alternated. Hence

$$\sum_{r=1}^m b_r = n - m.$$

Since that the number of 00 substrings in a string of $b_r + 1$ consecutive 0's is equal to b_r , in order to compute x_n we have to count two times (the first block can be of 0's or 1's) the number of non-negative integers solutions of

$$\sum_{j=1}^{\lceil m/2 \rceil} b_{2j-1} = \sum_{j=1}^{\lfloor m/2 \rfloor} b_{2j} = \frac{n - m}{2}$$

where $(n - m)/2$ is an integer. Therefore

$$x_n = 2 \sum_{n - m \text{ even}} \binom{(n - m)/2 + \lceil m/2 \rceil - 1}{\lceil m/2 \rceil - 1} \binom{(n - m)/2 + \lfloor m/2 \rfloor - 1}{\lfloor m/2 \rfloor - 1} = 2 \binom{n - 2}{\lfloor n/2 \rfloor - 1}.$$

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