Problem 11423
(American Mathematical Monthly, Vol.116, March 2009)

Proposed by G. Minton (USA).

Show that if n and m are positive integers with $n \geq m$ and $n-m$ even, then

$$\int_0^{+\infty} \frac{(\sin x)^n}{x^m}\, dx$$

is a rational multiple of π.

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If $m > 2$ then by integrating by parts

$$\begin{aligned}
\int_0^{+\infty} \frac{(\sin x)^n}{x^m}\, dx &= -\frac{1}{m-1}\int_0^{+\infty} (\sin x)^n\, d\left(\frac{1}{x^{m-1}}\right) \\
&= -\frac{1}{m-1}\left[\frac{(\sin x)^n}{x^{m-1}}\right]_0^{+\infty} + \frac{n}{m-1}\int_0^{+\infty} \frac{(\sin x)^{n-1}\cos x}{x^{m-1}}\, dx \\
&= -\frac{n}{(m-1)(m-2)}\int_0^{+\infty} (\sin x)^{n-1}\cos x\, d\left(\frac{1}{x^{m-2}}\right) \\
&= -\frac{n}{(m-1)(m-2)}\left[\frac{(\sin x)^{n-1}\cos x}{x^{m-2}}\right]_0^{+\infty} \\
&\quad + \frac{n}{(m-1)(m-2)}\int_0^{+\infty} \frac{(n-1)(\sin x)^{n-2}(\cos x)^2 - (\sin x)^{n-1}}{x^{m-2}}\, dx \\
&= \frac{n(n-1)}{(m-1)(m-2)}\int_0^{+\infty} \frac{(\sin x)^{n-2}}{x^{m-2}}\, dx - \frac{n^2}{(m-1)(m-2)}\int_0^{+\infty} \frac{(\sin x)^n}{x^{m-2}}\, dx.
\end{aligned}$$

Hence it suffices to consider the cases when $m = 1$, $n = 2a-1$, and $m = 2$, $n = 2a$ for some positive integer a.

$$\begin{aligned}
\int_0^{+\infty} \frac{(\sin x)^{2a-1}}{x}\, dx &= \frac{1}{2^{2a-2}}\sum_{k=0}^{a-1}(-1)^{a+k-1}\binom{2a-1}{k}\int_0^{+\infty} \frac{\sin((2a-2k-1)x)}{x}\, dx \\
&= \frac{1}{2^{2a-2}}\sum_{k=0}^{a-1}(-1)^{a+k-1}\binom{2a-1}{k}\int_0^{+\infty} \frac{\sin x}{x}\, dx \\
&= \frac{\pi}{2^{2a-1}}\sum_{k=0}^{a-1}(-1)^{a+k-1}\binom{2a-1}{k} = \frac{(2a-3)!!}{(2a-2)!!}\cdot\frac{\pi}{2}.
\end{aligned}$$

In a similar way it can be seen that

$$\int_0^{+\infty} \frac{(\sin x)^{2a}}{x^2}\, dx = n\int_0^{+\infty} \frac{(\sin x)^{2a-1}\cos x}{x}\, dx = \frac{(2a-3)!!}{(2a-2)!!}\cdot\frac{\pi}{2}.$$

□