Problem 11417
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Proposed by C. Lupu and T. Lupu (Romania).

Let f be a continuously differentiable real-valued function on $[0,1]$ such that $\int_{1/3}^{2/3} f(x)\,dx = 0$. Show that

$$\int_0^1 (f'(x))^2\,dx \geq 27\left(\int_0^1 f(x)\,dx\right)^2.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Let $G(x)$ be the piecewise differentiable function defined as

$$G(x) = \begin{cases} x & \text{if } x \in [0,1/3) \\ 1-2x & \text{if } x \in [1/3,2/3) \\ x-1 & \text{if } x \in [2/3,1) \end{cases}$$

then, since $\int_{1/3}^{2/3} f(x)\,dx = 0$

$$\int_0^1 f(x)\,dx = \int_0^{1/3} f(x)\,dx - 2\int_{1/3}^{2/3} f(x)\,dx + \int_{2/3}^1 f(x)\,dx = \int_0^1 f(x)G'(x)\,dx,$$

and by integrating by parts we obtain

$$\int_0^1 f(x)\,dx = \int_0^1 f(x)\,d(G(x)) = [f(x)G(x)]_0^1 - \int_0^1 G(x)\,d(f(x)) = \int_0^1 (-G(x))f'(x)\,dx.$$

Hence by Cauchy-Schwarz inequality

$$\left(\int_0^1 f(x)\,dx\right)^2 \leq \int_0^1 (G(x))^2\,dx \cdot \int_0^1 (f'(x))^2\,dx = \frac{1}{27}\int_0^1 (f'(x))^2\,dx.$$

Note that if $\int_{1/3}^{2/3} f(x)\,dx = 0$ then the equality holds if and only if $f'(x) = cG(x)$ for some real constant c. $\square$