Problem 11406
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Find

$$\sum_{i=0}^{n} \binom{n}{i} (2i-1)!!\,(2(n-i)-1)!!$$

in closed form.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Since $(2n-1)!! = (2n)!/(2^n n!)$ and

$$\sum_{i=0}^{n} \binom{2i}{i} \binom{2(n-i)}{n-i} = 4^n$$

then

$$\begin{aligned} \sum_{i=0}^{n} \binom{n}{i} (2i-1)!!\,(2(n-i)-1)!! &= \sum_{i=0}^{n} \frac{n!}{i!\,(n-i)!} \cdot \frac{(2i)!}{2^i\,i!} \cdot \frac{(2(n-i))!}{2^{n-i}\,(n-i)!} \\ &= \frac{n!}{2^n} \sum_{i=0}^{n} \binom{2i}{i} \binom{2(n-i)}{n-i} = n!2^n = (2n)!!\,. \end{aligned}$$

$\square$