Problem 11403
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Proposed by Yaming Yu (USA).

Let n be an integer greater than 1, and let f_n be the polynomial given by

$$\sum_{i=0}^{n} \binom{n}{i} (-x)^{n-i} \prod_{j=0}^{i-1} (x+j).$$

Find the degree of f_n.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Let $\left[{i \atop k}\right]$ be the Stirling number of first kind which counts the number of permutations of $\{1, 2, \ldots, i\}$ with k-cycles. Since

$$\prod_{j=0}^{i-1} (x+j) = x^{\bar{i}} = \sum_{k=0}^{i} \begin{bmatrix} i \\ k \end{bmatrix} x^k$$

then the coefficient of x^m in f_n is

$$a_{n,m} = \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} \begin{bmatrix} i \\ m-(n-i) \end{bmatrix} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \begin{bmatrix} n-k \\ m-k \end{bmatrix}.$$

Note that $\binom{n}{k}\left[{n-k \atop m-k}\right]$ is the number of permutations of $\{1, 2, \ldots, n\}$ which have m cycles with at least k fixed points (a fixed point is a cycle of length 1): $\binom{n}{k}$ is the number of ways of choosing k fixed points and $\left[{n-k \atop m-k}\right]$ is the number of ways of completing the permutation with other $m-k$ cycles. Hence the formula of $a_{n,m}$ can be seen as an application of the Inclusion-Exclusion Principle and it follows that $a_{n,m}$ is the number of permutations of $\{1, 2, \ldots, n\}$ with m cycles of length greater than 1. So, $a_{n,m} = 0$ when $2m > n$ that is if $m > \lfloor n/2 \rfloor$ and $a_{n,m} > 0$ when $m = \lfloor n/2 \rfloor$ because

$$(1,2)(3,4)\ldots(n-1,n) \text{ for } n \text{ even and } (1,2)(3,4)\ldots(n-2,n-1,n) \text{ for } n \text{ odd}$$

have $\lfloor n/2 \rfloor$ cycles of length greater than 1.

Hence the degree of f_n is $\lfloor n/2 \rfloor$. □